DEPARTMENT OF EDUCATION

GRADE 12 MATHEMATICS A

12.1: PATTERNS AND ALGEBRA

FODE DISTANCE LEARNING

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GRADE 12

MATHEMATICS A

UNIT MODULE 1

PATTERNS AND ALGEBRA

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Principal-FODE

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Written by: Luzviminda B. Fernandez
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CONTENTS

SECRETARY'S MESSAGE

Achieving a better future by individuals students, their families, communities or the nation as a whole, depends on the curriculum and the way it is delivered.

This course is part and parcel of the NDOE new reformed curriculum. Its learning outcomes are student centred and written in terms that allow them to be demonstrated, assessed and measured.

It maintains the rationale, goals, aims and principles of the National Curriculum and identifies the knowledge, skills, attitudes and values that students should achieve.

This is a provision of Flexible, Open and Distance Education as an alternative pathway of formal education.

The Course promotes Papua New Guinea values and beliefs which are found in our constitution, Government policies and reports. It is developed in line with the National Education Plan (2005 – 2014) and addresses an increase in the number of school leavers which has been coupled with a limited access to secondary and higher educational institutions.

Flexible, Open and Distance Education is guided by the Department of Education's Mission which is fivefold:

- to facilitate and promote the integral development of every individual
- to develop and encourage an education system which satisfies the requirements of Papua New Guinea and its people
- to establish, preserve, and improve standards of education throughout Papua New Guinea
- to make the benefits of such education available as widely as possible to all of the people
- to make education accessible to the physically, mentally and socially handicapped as well as to those who are educationally disadvantaged

The College is enhanced to provide alternative and comparable pathways for students and adults to complete their education, through one system, many pathways and same learning outcomes.

It is our vision that Papua New Guineans harness all appropriate and affordable technologies to pursue this program.

I commend all those teachers, curriculum writers and instructional designers, who have contributed so much in developing this course.

....

DR. UKE KOMBRA, PhD
Acting Secretary for Education

UNIT 1: PATTERNS AND ALGEBRA

Introduction

Algebra is a language of symbols and patterns. The study of patterns is a key part of algebraic thinking. It is important that you, as students, are able to recognize and analyse symbols and patterns and make generalizations about them. Working with symbols and number patterns helps you develop facility, flexibility and familiarity with numbers as well as build understanding of general number properties. Work with patterns also allows you to develop logical reasoning skills, make conjectures, and test your ideas about them.

This unit module was created to help students realize the importance of learning Patterns and Algebra as a universal "brain builder" along with preparing them for careers in mathematics, business and science. Basic terminologies are defined and concepts are discussed in a simplified manner with substantial illustrations. Problem solving examples covers different disciplines of applied mathematics such as engineering, physical and natural sciences, business and accounting.

This unit focuses on different methods of solving real-life problems through the following topics – sets, sequences and series, binomial theorem and determinants. Learning activities are provided in almost every sections of the topic of a unit.

LEARNING OUTCOMES

On successful completion of this module, you will be able to:

- identify the methods of describing set
- determine finite and infinite sets
- apply universal set and compliment of a set
- draw and analyse Venn diagram
- find the n^{th} terms in arithmetic and geometric progressions
- find the sum of the n^{th} terms in arithmetic and geometric progressions
- find the sum of a series in AP by using summation notation
- apply the rules of arithmetic and geometric progressions to solve practical problems in real life
- find the limiting sum of a geometric progression
- use the binomial theorem to expand binomial expression
- expand a binomial expression using the Pascal's triangle
- find the solutions of system of equation using determinants
- use the Cramer's rule to solve determinants of order two.

TIME FRAME

This unit should be completed within 10 weeks.

If you set an average of 3 hours per day, you should be able to complete the unit comfortably by the end of the assigned week.

Try to do all the learning activities and compare your answers with the ones provided at the end of the unit. If you do not get a particular exercise right in the first attempt, you should not get discouraged but instead, go back and attempt it again. If you still do not get it right after several attempts then you should seek help from your friend or even your tutor. Do not pass any question without solving it first.

12.1.1 Sets

A very important concept in any branch of mathematics is that of **sets**. The study of set was originally developed by Georg Cantor (1845-1918) as a means of investigating the theory of infinite series. We often deal with groups or collection of objects, such a set of books, a group of students, a list of states in a country, a collection of baseball cards, etc.

Sets may be thought of as a mathematical way to represent collections or groups of objects. The concept of sets is an essential foundation for other topics in mathematics.

This section covers the essential concepts of math set theory. This includes the basic ways of describing sets, use of set notation, finite sets, infinite sets, empty sets, subsets, universal sets, complement of a set, basic set operations including intersection and union of sets, use of Venn diagrams and simple applications of sets.

12.1.1.1 Sets and Set Notation

We begin this topic by discussing sets and set notation.

Recall from your discussion of sets (Grade 7 Mathematics Strand 5) that a set is a well-defined collection of objects. The following are just some examples of a set:

a. The set of past Prime Ministers of Papua New Guinea.
b. The set of Papua New Guinea colleges and universities.
c. The set of students in mathematics whose names start with the letter "R".
d. The set of distinct letters of the word "algebra".
e. The set of rational numbers between 99 and 100.
f. The set of shapes that tessellate.
g. The set of negative numbers.
h. The set of all the factors of 150.
i. The set of reciprocals of the numbers 1, 2, 3 and 4.

It should be emphasized that a set has to be "well-defined' so that there is no question as to whether a certain object belongs or does not belong to the set.

An object which belongs to a set is called **element** or **member** of the set. Capital letters are usually used to denote the name of a set while a small letters are used for its members.

The symbol $\in$ denotes membership while $\notin$ denotes non-membership to a set.

Thus, "**a ∈ A**" (read as "**a** is an element of **A**") indicates that the object **a** is an element of the set **A** while "$a \notin$ A" (read as "**a** is not an element of **A**") means that **a** does not belong to **A**.

Examples of identifying elements of a set

Insert the symbol ∈ or ∉ in the space provided to make each statement true.

i. 1 _____ {numbers that are their own reciprocals}

Answer: 1 ∈ {numbers that are their own reciprocals}
Since **1 is equivalent to** $\frac{1}{1}$

ii. Broccoli _____ {red vegetables}

Answer: Broccoli ∉ {red vegetables}
Since **Broccoli is not a red vegetable**

iii. ○ _____ {polygons}

Answer: ○ ∉ {polygon}
Since **Circle is not a polygon**

There are three methods of describing a set.

1. **Roster Method or List method** – listing the elements of a set inside a pair of braces { } with any two elements separated by a comma.

Examples

a. Set A = {1, 2, 3, 4, 5}

b. Set B = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

c. Set C = {Ampicillin, Amoxicillin, Penicillin, Cepalexin}

d. Set D = {January, February, March, April, May, June, July, August, September, October, November, December}

2. **Modified Roster Method**- sometimes the number of elements in a set is so large that it is not convenient or even possible to list all its members. In such case we modify the roster notation.

Examples

a. Set X = {The set of natural numbers} can be represented as follows:

Set X = {1, 2, 3, 4, ...}

This is read "The set whose elements are 1, 2, 3, 4**, and so on.**"

The three dots to the right of the number 4 indicate that the remaining numbers are to be found by counting in the same way we have begun: namely by adding 1 to the preceding number to find the next number.

b. Set Y = {The set of all even numbers}

Set Y = {2, 4, 6, 8, ...}

This is read "The set whose elements are 2, 4, 6, 8 **and so on.**"

The three dots to the right of the number 8 indicate that the remaining numbers are to be found by counting in the same way we have begun: namely by adding 2 to the preceding number to find the next number.

3. **Rule Method or Defining Property Method** – describing the elements of the set wherein the criteria for membership in the set are given. It usually uses the **set builder notation** wherein the symbol **x** is used to represent any member of the given set.

Examples

The examples in number 1 can also be written as follows:

a. Set A = $\{x | x \leq 5, x \in N\}$. This is read as "the set of all x, such that x is a counting number less than or equal to 5.

A shorter and more efficient notation is

Set A = {x|x is a counting number less than or equal to 5}

where the vertical bar is read "**such that**".

b. Set B = {x|x is a day of the week}

Now let us have examples of changing from set-builder notation to roster notation.

Example 1

Write {x|x + 2 = 5 and x is a natural number} in roster notation. (The digits are: 0, 1, 2, 3, 4, 5, 6 7, 8, 9.)

Answer: {x|x is an odd number} = {1, 3, 5, 7, 9}

Example 2

Write {x|x is an odd number} in roster notation.

Answer: {x|x + 2 = 5 and x is a natural number} = {3}

A set is considered **finite** if one could probably enumerate all the elements of the set. Otherwise, it is said to be **infinite**.

The set that does not contain any element is called the **empty set** (or **null set**), denoted by the symbols **{ }** or **Ø**. And a set containing all the elements under consideration is called the **universal set**, denoted by the capital letter **U**.

Examples of identifying finite and infinite sets

i. The set of days in a week is finite.

ii. The set of counting numbers greater than 8 is infinite.

iii. The set of all rivers on the earth is finite (even if it may be difficult to count them).

iv. {Letters of the alphabet} is finite. There are exactly 26 elements in the set.

v. {Rational numbers between 99 and 100} is infinite. There are an infinite number of rational numbers between any two rational numbers.

vi. A = {5, 9, 10, 13} is finite.

vii. N = {1, 2, 3, ...} is infinite.

ix. The set of digits is finite.

x. The set of all fractions is infinite.

Now do the learning activity on the next page.

Learning activity 12.1.1.1

20 minutes

1. Use both the roster and the rule method to specify the following sets.

   a. The counting numbers greater than 4 but less than or equal to 10.

   b. The distinct letters of the word "mathematics".

   c. The counting numbers less than 100 and ending in 0.

2. For each of the following sets, write whether it is finite, infinite or empty.

   a. The set of counting number between 5 and 27.

   b. The set of distinct letters of the word "patrimony".

   c. The set of even prime numbers.

   d. The set of Papua New Guinean heroes.

   e. The set of real numbers which satisfies the equation $x^2 = 4$.

12.1.1.2 Set Relations

Two sets A and B are said to be **equal**, denoted by A = B, if and only if they have identical (exactly the same) elements. Otherwise, they are **unequal**.

Examples of equal sets

1. If A = {1, 3, 5, 6, 8}; B = {3, 6, 1, 8, 5} and C = { 6, 3, 5, 8, 1} then A, B and C are equal. Notice that all the sets have exactly the same elements, even though they are not listed in the same order.
2. If A = {1, 5, 5, 5} and B = {5, 1}, then A and B are equal. Notice that both sets have exactly the same elements. It is not necessary to write the same element more than once when writing the roster of a set.

Notice from these examples that a set does not change if its elements are simply rearranged or repeated.

Examples of unequal sets

1. If A = {7, 9, 11} and {7, 11}, then A and B are not equal sets because they both do not have exactly the same elements.

A **"one-to-one correspondence"** between two sets, A and B, is a pairing between the elements of A with the elements of B in such a way that each element of A is paired with one and only one element of B and each element of B is paired with one and only one element of A.

Two sets A and B are said to be **equivalent,** denoted by A $\leftrightarrow$ B or A $\approx$ B, if and only if there exists a one-to-one correspondence between A and B.

Example: If A ={c, a, t} and B = {d, o, g} then the sets A and B are **equivalent.**

A = {c, a, t}

B = {d, o, g}

Note that equal sets are equivalent but equivalent sets are not necessarily equal.

Example: If **A** = {pawpaw, mango, banana} and **B** = {apple, grapes, strawberry},

then the sets **A** and **B** are equivalent.

Two sets **A** and **B** are said to be **disjoint** if they do not have common elements.

Example: If we let **A** = {peter, John, Paul, Matthew} and **B** = {Thomas, Isaac, Job}, then the sets **A** and **B** are disjoint.

Set **A** is said to be a **subset** of another Set **B**, denoted by **A** $\subset$ **B** (read as "**A** is a subset of **B**" or "**A** is contained in **B**" or "**B** contains **A**") if all the elements of **A** are also elements of **B**.

Otherwise, if **A** has at least one element that is not an element of **B**, then **A** is not a subset of **B**, denoted by **A** $\not\subset$ **B**.

Examples

1. Consider the sets **A** = {1, 3, 5}, **B** = {2, 4, 6} and **C** = {1, 2, 3, 4, 5}.

   Set **A** is subset of **C** because all elements of A are elements of C.
   In symbol, we write **A** $\subset$ **C.**

   Set **B** is not a subset of **C** because not all the elements of B are in C.
   In symbol, we write B $\not\subset$ C.

Notes:

- The null set **Ø** is considered to be a subset of every set, say Set **A**. This is true because **Ø** is empty and therefore we cannot find an element in **Ø** that does not belong to **A**.
- Any set is a subset of itself because all the elements of a set are found in the set itself.
- For any set **A**, the null set **Ø** and the set **A** itself are **trivial sets** of **A**. This means that:
  - ♣ An empty set is always a subset of every set.
  - ♣ Every set is always a subset of itself.
- We call another set, let us say Set **B**, a **proper subset** of **A**, if **B** is a subset of **A** and **B** is not equal to **A**. In other words, **B** is said to be a proper subset of **A** if all the elements of **B** are found in **A** but there are some elements in **A** that are not found in **B**.
- Two sets **A** and **B** are equal if and only if **A** $\subset$ **B** and **B** $\subset$ **A**.

Now do the learning activity on the next page.

Learning Activity 12.1.1.2

10 minutes

Answer the following:

1. Let A = {x | 2x = -8} and b = -4. Is A = b?

2. Which of the following sets are equal?

   A = {x | x is a letter in the word "read"}

   B = set of distinct letters in the word "dreaded"

   C = {x | x is a letter in the word "reader"}

   D = {d, a. r, e}

3. Determine whether the first set is a subset of the second set. Use the correct symbol.

   a. E = {even numbers}
      R = {real numbers}

   b. P = {parallelograms}
      S = {squares}

   c. C = {counting numbers}
      P = {prime numbers}

4. If A = {a b, c} and B = {a, b, c, d}, which of the following is true?

   a. $A \in B$
   b. $A \subset B$
   c. $a \in A$
   d. $b \in B$
   e. $b \subset B$
   f. $\emptyset \in B$

5. For any set A, its subsets are the following except

   A. the set itself.
   B. the null set.
   C. the universal set.
   D. all proper subsets of A.

6. The statement "equal sets are equivalent" is

   A. always true
   B. never true
   C. sometimes true
   D. cannot be determined

7. Write the following from set-builder notation to roster notation.

   a. A = {x|x is an odd digit}

   b. B = {x|x + 2 =5 and x is a natural number}

8. Write the following from roster notation to set-builder notation.

   a. A = {2, 4, 6, 8, ...}

   b. B = {5, 10, 15, 20,...}

12.1.1.3 Sets Operations

We could also define operations on sets such as union, intersection, complementation and cross product and set difference in the same manner that we could perform basic operations such as addition, subtraction, multiplication and division on numbers.

Union of Sets

The **union** of two sets **A** and **B**, denoted by **A∪ B** (read "**A** union **B**"), is the set of all elements that are in **A** or in **B**.

$$\boxed{A \cup B = \{x \mid x \in A \text{ or } x \in B\}}$$

Example 1

If A = {1,2,3,4,5,6} and B= { 2,4,6,8,10,12},

then A∪ B = {1, 2, 3, 4, 5, 6, 8, 10, 12}.

Example 2

If A = {d, r, u, m} and B = {d, r, e, a, m},

then A∪ B = {d, r, u, e, a, m}.

Example 3

If Z = {0, 1, 2, 3} and T = {2, 3, 4, 5},

then Z∪ T = {0, 1, 2, 3, 4, 5}.

Example 4

If R = {2, 4, 6, 8, 10, 12} and S = {1, 2, 3, 4, 5}

then $R \cup S$ = {1, 2, 3, 4, 5, 6, 8, 9, 10, 12}

Example 5

If C = { x, y} and D = { 4, 5}, then $C \cup D$ = {x, y, 4, 5}

Notes:

1. Set union is commutative. Thus $A \cup B = B \cup A$
2. Sets A and B are always subsets of their union. $A \subset A \cup B$ and $B \subset A \cup B$.

Intersection of Sets

The **intersection** of two sets A and B, denoted by $A \cap B$ (read "A intersection B"), is the set of all elements that are common to both sets.

$$\boxed{\mathbf{A \cap B = \{x \mid x \in A \text{ or and } x \in B\}}}$$

Example 1

If A = {d, r, u, m} and B = {d, r, e, a, m}, then $A \cap B$ = {d, r, m}.

Example 2

If C = {2, 4, 6, 8} and D = {1, 2, 3, 4, 5, 6} and E = {3, 6, 9}, then $C \cap D \cap E$ = {6}.

Example 3

If E = {a, 6, b} and F = {c, 5, 4}, then $E \cap F$ = { }

Notes:

1. Set intersection is commutative. Thus $A \cap B = B \cap A$.
2. Each of Set A and B contains their intersection as a subset. $(A \cap B) \subset A$ and $(A \cap B) \subset B$.
3. If Sets A and B are disjoint, then $A \cap B \subset A = \emptyset$

Difference of Two Sets

The **difference** of two sets A and B, denoted by A – B (read "A minus B") is a set of all elements which belongs to A but which do not belong to B.

$$\mathbf{A - B = \{x \mid x \in A \text{ and } x \in B\}}$$

Example 1: If we let A = {d, r, u, m} and let B = {d, r, e, a, m},

then A – B = {u} and B – A = {e, a}

Example 2: If Set A = {2, 4, 7, 11} and Set B = {4, 7, 12}

then, A – B = {2, 11} AND B – A = {12}

Notes:

1. Set difference is not commutative. Thus $A - B \neq B - A$
2. Set A contains A – B as a subset. $(A - B) \subset A$
3. The sets (A – B), $A \cap B$ and (B – A) are mutually disjoint, that is the intersection of any two of these sets is the null set.

Complement of a Set

The **complement** of a set A, denoted by A' (read "A complement") is the set of all elements in the universal set which is not in the given set A.

$$\mathbf{A' = \{x \mid x \notin A \text{ and } x \in U\}}$$

Example 1: Let U = {letters of the English alphabet} and A = {a, b, c, x, y, z}

Then A' = {d, e, f, g, ..., u, v, w}

Example 2: Let U = {all digits} and B = { 0, 1,3, 5, 7}

Then B' = {2, 4, 6, 8, 9}

Notes:

1. The union of any set and its compliment is the universal set. $A \cap A' = U$.
2. Any set and its complement are disjoint. $A \cap A' = \emptyset$
3. The complement of the universal set is the null set. $U' = \emptyset$
4. The complement of the null set is the universal set. $\emptyset' = U$

5. The complement of the complement of a set is the set itself. $(A')' = A$
6. The difference of A and B is equal to the intersection of A and the complement of B. $A - B = A \cap B'$
7. For any two sets A and B, $(A - B) \cap B = \emptyset$
8. F or any two sets A and B, $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

Set Product or Cross product of Two Sets

The **set product or cross product** of two sets A and B, denoted by **A** x **B** (read "**A** cross **B**"), is the set of all possible ordered pairs (a, b), where **a** is any element of **A** and **b** is any element of **B**.

$$\boxed{\mathbf{A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}}}$$

Example 1

Let **A** = {a, b} and **B** = { x, y, z}.

Then **A x B** = {(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)}

while **B x A** = {(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)}

Example 2

Let **M** = {2, 3} and **N** = {n, i, k, e}.

Then **M x N** = {(2, n), (2, i), (2, k), (2, e), (3,n), (3,i), (3, k) (3,e)}.

Note:

1. Set product or cross product is not commutative. **A x B** $\neq$ **B x A**, unless **A** = **B** or one of the factors is empty.
2. For any set **A**, **A x** $\emptyset$ **=** $\emptyset$ **x A =** $\emptyset$

Now do the learning activity on the next page.

Learning Activity 12.1.1.3

20 minutes

1. Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, and

   A = {1, 3, 4, 5. 6}; B = {4, 5, 9, 10}; C = {1, 2, 3, 7, 8}

   Find the following:

   a. A'

   b. B'

   c. C'

   d. $A \cup B$

   e. $A \cup C$

   f. $A \cap B$

   g. $A - B$

   h. $B \times C$

   i. $A' \times B'$

   j. $(A - C) \times (A \cap B)$

2. Let U = {x | x is a positive integer less than 20}, and its subsets.

   A = {x | x is a positive prime number less than 20}.

   B = {x | x is a positive divisor of 18}.

   C = {x | x is a positive divisor of 12}.

   Find the following:

   a. $A' \cup B'$

   b. $A - C$

c. B – A

d. $A \cap B \times B \cap C$

3. Given: A = {a, 5, 7, b}, B = {c, 6, 7, b} and C = {3, 4, 6}
Find:

a. $A \cup B$

b. $C \cup A$

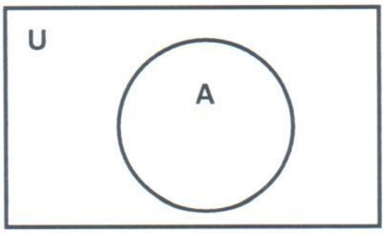
c. $A \cap B$

d. $A \cap C$

12.1.1.4 Venn diagram

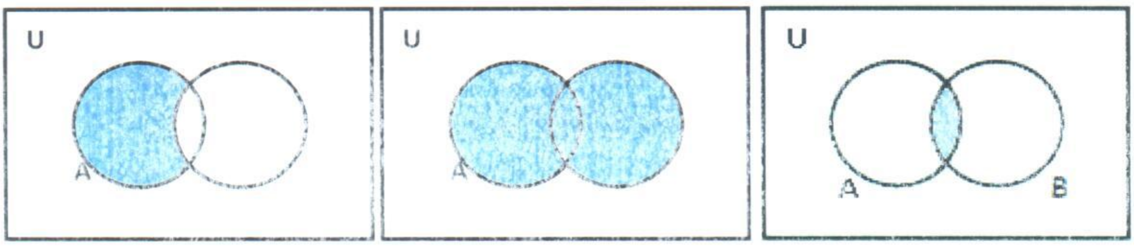
A very useful pictorial representation in dealing with the relation and operation of sets is the **Venn diagram.**

Shown below is a simple Venn diagram.



The universal set U is represented by a rectangle and all its subsets are drawn as circles within the rectangle. When two circles overlap, the region shared by both circles represents the intersection of the two sets.

For example, the shaded portion in the following diagrams shows the caption below the figure.



A – B $\quad$ $A \cup B$ $\quad$ $A \cap B$

Drawing Venn Diagrams

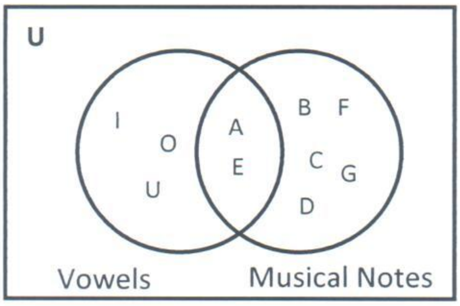
To draw a Venn diagram, you have to determine the relationships between the sets.

Example 1

Draw the Venn diagram to show the relationship between the given sets.

Vowels = {A, E, I, O, U}
Letters to represents musical notes = {A, B, C, D, E, F, G}

Solution: The intersection of the sets = {A, E}

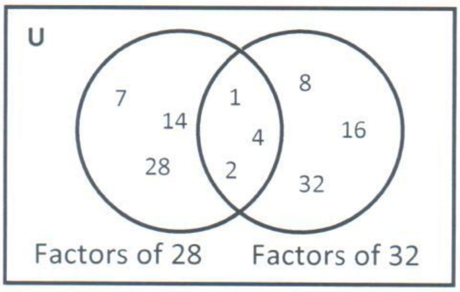


Example 2

Draw the Venn diagram to show the relationship between the given sets.

Factors of 28 = {1, 2, 4, 7, 14, 28}
Factors of 32 = {1, 2, 4, 8, 16, 32}

Solution: The intersection of the sets = {1, 2, 4}



Example 3

Given U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} with the following subsets:

A = {2, 4, 6, 8, 10}

B = {1. 4, 9}

C = {2, 6, 8}

Then, $A \cup B = \{1, 2, 3, 4, 6, 8, 9, 10\}$

$B \cap C = \{\emptyset\}$

$A \cup C = \{1, 2, 4, 6, 8, 9\}$

$A' = \{1, 3, 5, 7, 9\}$

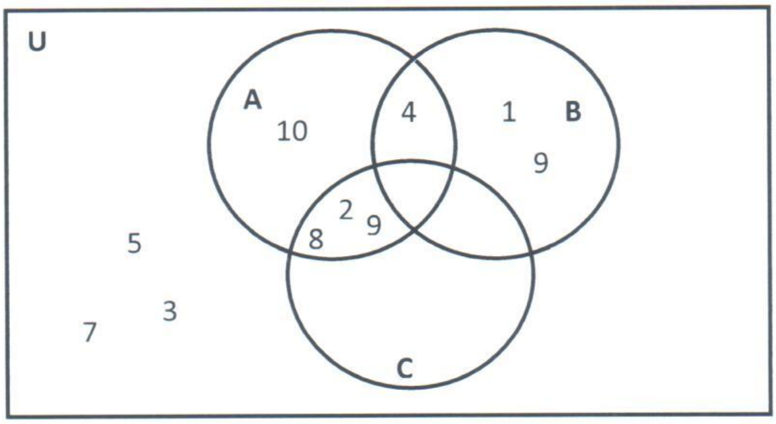
$A \cap B = \{4\}$

$B' = \{2, 3, 5, 6, 7, 8, 10\}$

$A \cap C = \{2, 6, 8\}$

$(A \cup B)' = \{3, 5, 7\}$

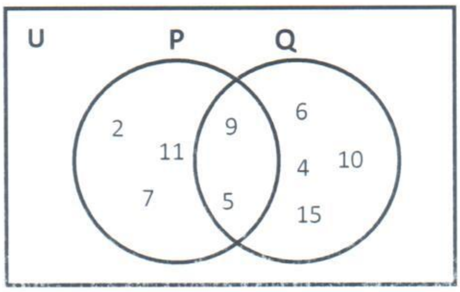
The following Venn diagram shows the relationships among the given sets.



Venn diagram can also be used to identify intersections, unions and subsets.

Example 4

Below is a Venn diagram. Identify the intersection, union and subset.



Solution: $P \cap Q = \{5, 9\}$

$P \cup Q = \{2, 4, 5, 6, 7, 8, 9, 10, 11, 15\}$

Subset: none

Venn diagram can also be used to analyse survey problems.

Example 5

The career references of a group of 60 students were recorded as follows:

35 students prefer medical careers (**M**)
45 students prefer careers in engineering (**E**)
40 students prefer science-related careers (**S**)
18 would like to be consider careers in both medicine and sciences
20 would like to consider careers in both engineering and sciences
25 would like to consider careers in both engineering and medicine
10 would consider all these career paths

These data above can be summarised and easily analysed by means of a Venn diagram

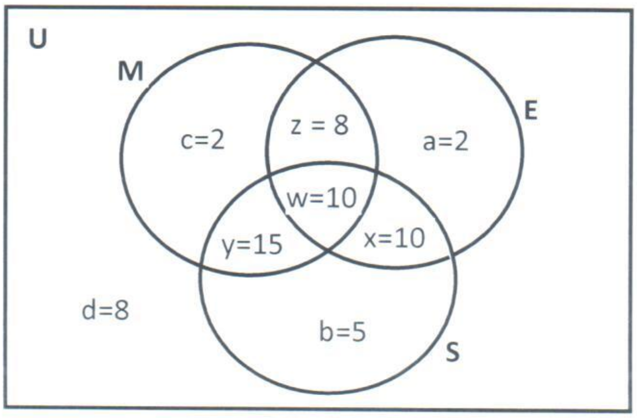
To present the data in a Venn diagram, we let the universal set **U** to contain all the 60 students who were surveyed. And we use three circles to represent the three career options namely medicine (**M**), engineering (**E**) and Science (**S**). We first put in the 10 students who would consider all the 3 career options. This is the region marked **w**. Since there are 20 students who like careers in both engineering and science, we must have 20 – 10 = 10 who prefer engineering and science but not medicine. We marked this region **x**.

Similarly, there are 25 – 10 = 15 students who prefer science and medicine but not engineering. We marked this region **y**. And there are 18 – 10 = 8 students who prefer engineering and medicine but not science. We marked this region **z**.

Based on the data, there are 30 students who prefer engineering. But if we look at the circle for engineering we see that we have accounted for **w** + **x** + **z** = 28 students. This leaves 30 – 28 = 2 students who prefer engineering only (region **a**).

Similarly, we have 40 – 10 – 10 – 15 = 5 students who prefer science only (region **b**) and we have 35 – 10 – 15 – 8 = 2 students who prefer medicine only (region **c**). Finally we fill in region marked **d** with the remaining 8 students who do not like any of these three careers to bring the total to the given total number of 60 students.

Here is the Venn diagram.



Using the Venn diagram on the previous page, we can also easily verify the following additional results from the survey.

a. The number of students who do not like any of the three careers.

$d = 60 - (a + b + c + w + x + y + z)$

$d = 60 - (2 + 5 + 2 + 10 + 10 + 15 + 8)$

$d = 60 - 52$

$d = 8$

b. The number of students who prefer medicine only. $c = 2$

c. The number of students who prefer exactly one of the three careers.

$a + b + c = 2 + 5 + 2 = 9$

d. The numbers of students who prefer exactly two of the three careers.

$x + y + z = 10 + 15 + 8 = 33$

e. The number of students who do not like medicine.

$a + b + d + x = 2 + 5 + 8 + 10 = 25$

Now do the learning activity.

Learning Activity 12.1.1.4

20 minutes

1. Given: U = {months of the year} and its subsets

A = {months which ends with "ber"}

B = {months with exactly 30 days}

C = {first or fourth quarter month}

Draw a Venn diagram showing the relationships among the given sets above and list elements of the following sets.

a. $B \cup C$

b. $B \cap C$

c. $A - C$

2. In a recent survey of 300 housewives, the following were determined with regard to their washing detergent preferences.

105 use Klin	40 use Klin and Sudso
100 use Sudso	50 use Klin and Cold Power
120 use Cold Power	18 use Klin and Cold Power and Sudso
35 use Sudso and Cold power	

a. Draw a Venn diagram to summarize the given data.

b. Determine the number of housewives who use:

i. Klin only

ii. Sudso only

iii. Cold Power only

c. Determine the number of housewives who do not use any of the three washing detergents.

d. Number of housewives who prefer one detergent only

e. Number of housewives who prefer exactly two of the three detergents

NOW DO THE SUMMATIVE TASK 1

SUMMATIVE TASK 12.1.1

30 minutes

Answer the following questions.

1. Is the collection consisting of Δ, X, $\subseteq$, $\sum$, and 6 a set?

2. Are {3, 3, 8, 8} and {8, 3} equal sets?

3. Write all the elements in the set of digits.

4. Which of the following sets are finite and which are infinite?

   a. The set of whole numbers Answer: __________

   b. The set of books in the National Library Answer: __________

   c. The set of days in a week Answer: __________

   d. The set of all college students in Port Moresby today Answer: __________

5. Which of the following statements is true and which is false?

   a. If A = {5, 12, 20}, then $12 \in A$. Answer: __________

   b. If B = {w, x, y, z}, then $a \notin B$. Answer: __________

   c. $0 \in \emptyset$ Answer: __________

   d. $\emptyset \notin \{\ \}$ Answer: __________

6. Write {x | x is an even digit} in roster notation.

7. Write {5, 10, 15,...} in set-builder notation.

8. Given A = {1, 2, 3, 4} and B = {2, 4, 5}

   Find: (a) $A \cup B$

   (b) $A \cap B$

9. Choose the correct answer to each question from the alternatives A, B, C or D. Circle the letter of your choice.

   i. For any Set X, its subsets are the following EXCEPT

   A. set itself B. null set

   C. universal set D. all proper subset of Set A

ii. The statement “equal sets are equivalent“ is

A. always true B. never true

C. sometimes true D. cannot be determine

iii. If $X \subset Y$, then $(X \cup Y) \cap (Y \cap X)$ is equal to

A. X B. Y

C. $\emptyset$ D. U

iv. The following statements about any set X are true except

A. $X \subset X$ B. $X \cap U = U$

C. $X \cap \emptyset = \emptyset$ D. $X \cap X' = U$

v. If X and Y are disjoint sets, then $X' \cup (Y \cap X)$ is equal to

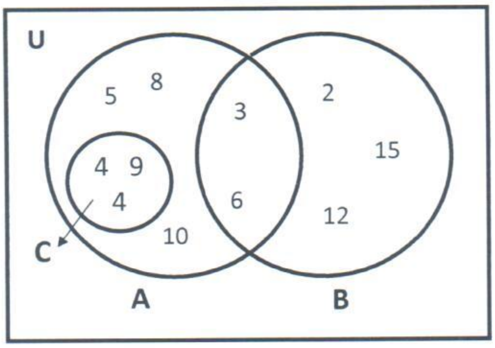
A. X' B. X

C. $\emptyset$ D. Y

10. Draw a Venn diagram on the space below to show the intersection and union of the given sets.

Set	Elements
Ron’ Favorite TV channels	2, 4, 5, 6, 7 8
Eve’s Favorite TV channels	4, 6, 7, 9, 10, 14

11. Study the Venn diagram below.



Identify:

a. $A \cap B$ =

b. $A \cap C$ =

c. $B \cap C$ =

d. $A \cup B$ =

e. $B \cup C$ =

f. Subset of A =

12. Problem solving

i. Given: U = {months of the year} and its subsets

A = {months which ends with "ber"}

B = {months with exactly 30 days}

C = {second or third quarter month}

Draw a Venn diagram showing the relationships among the given sets above and list the elements of the following sets.

a. $B \cup C$

b. $B \cap C$

c. $A - C$

d. $(B \cap C)'$

e. $A' - B$

ii. A morning daily reported the results of a newspaper preference survey made on 200 subscribers.

Newspaper	Number of subscribers
A	80
B	65
C	85
A and B	40
A and C	35
B and C	20
All three	15
None of the three	10

Use a Venn diagram to check the accuracy of the data.

12.1.2 Sequences and Series

Many real-world processes generate lists of numbers. For instance, the balance in a bank account at the end of each month forms a list of numbers when tracked over time. Mathematician call such lists **sequences**.

Often, in mathematical work, groups of three or more numbers are encountered that are arranged in a definite order or sequence, so that each number of the group may be obtained from one or more of the preceding numbers by some fixed law. Such a group or sequence is called a **series.**

Of the many kinds of sequences series in this section we shall discuss only the two simplest namely, arithmetic and geometric. We shall also discuss summation notation.

12.1.2.1 Basic Concepts

A collection of numbers arranged in a particular order is called a **sequence**. This is also known as **progression.**

The following are examples of sequences.

1. $1, 3, 5, 7, 9, \ldots$
2. $-1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{16}, \ldots$

Three dots are used to indicate that the pattern continues. The numbers in the sequence are called **terms** of the sequence. If the sequence has last term, it is called **finite** and sequences which are not finite are called **infinite.** The two sequences above are infinite sequences while

$$2, 4, 6, 8, 10$$

is finite with the last term 10. For some sequences, it is possible to find a formula that describes a general term, called the **n^{th} term**.

The n^{th} term of the preceding sequence is 2n, where n indicates the particular term.

n	1	2	3	4	5
n^{th} Term (2n)	$2 \cdot 1$	$2 \cdot 2$	$2 \cdot 3$	$2 \cdot 4$	$2 \cdot 5$

Sometimes we used a letter with a subscript to indicate a particular term of a sequence.

a_1 = first term a_2 = second term a_3 = third term

and in general a_n = n^{th} term.

Thus, in our example $a_1 = 2$ $a_2 = 4$ $a_3 = 6$ $a_4 = 8$ $a_5 = 10$ and $a_n = 2n$.

To find the 103rd term of this sequence, we let n = 103 to get $a_{103} = 2(103) = 206$

> The n^{th} term of a sequence is also called the **general term** of a sequence. The formula for the general term is used to generate a sequence by repeated substitution of counting numbers.

Now look at the examples of finding the terms of a sequence.

Example 1

Write the first five terms of the infinite sequence with general term $a_n = 2n - 1$.

Solution:

1	2	3	4	5	. . .	n
$a_1 = 2(1) - 1$	$a_2 = 2(2) - 1$	$a_3 = 2(3) - 1$	$a_4 = 2(4) - 1$	$a_5 = 2(5) - 1$	. . .	$a_n = 2(n) - 1$
$= 1$	$= 3$	$= 5$	$= 7$	$= 9$		

Thus, the first five terms are **1, 3, 5, 7, 9**.

The sequence is **1, 3, 5, 7, 9, . . . 2n – 1, ...**

Example 2

A finite sequence has four terms, and the formula for the n^{th} term is $x_n = (-1)^n \frac{1}{2^{2-1}}$.

What is the sequence?

Solution:

When n = 1 $\quad x_n = x_1 = (-1)^1 \frac{1}{2^{2-1}} = (-1) \frac{1}{2^0} = -1$

When n = 2 $\quad x_n = x_2 = (-1)^2 \frac{1}{2^{2-1}} = (-1)^2 \frac{1}{2^1} = \frac{1}{2}$

When n = 3 $\quad x_n = x_3 = (-1)^3 \frac{1}{2^{3-1}} = (-1)^3 \frac{1}{2^2} = -\frac{1}{4}$

When n = 4 $\quad x_n = x_4 = (-1)^4 \frac{1}{2^{4-1}} = (-1)^4 \frac{1}{2^3} = \frac{1}{8}$

Thus, the sequence is $\quad -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}$

Example 3

Find the first four terms of a sequence whose n^{th} term is given by the formula $a_n = \frac{2n-1}{2n}$.

Solution: when n = 1 $\quad a_n = x_1 = \frac{2(1)-1}{2(1)} = \frac{1}{2}$

when n = 2 $\quad a_n = x_2 = \frac{2(2)-1}{2(2)} = \frac{3}{4}$

when n = 3 $\quad a_n = x_3 = \frac{2(3)-1}{2(3)} = \frac{5}{6}$

when n = 4 $\quad a_n = x_4 = \frac{2(4)-1}{2(4)} = \frac{7}{8}$

We are often interested in determining sequence, given a formula for the general term or n^{th} term, as in the previous examples. Also, the first terms of the sequence may be given, and we must find a formula for the general or n^{th} term. Usually the third type of problem is much more difficult to solve than the first and second. We shall only find the formula for a general term for two specific sequences, arithmetic and geometric sequences or progressions. We will be discussing these two in detail in the following sections.

Associated with every sequence is a **series**. It is the indicated sum of the terms of a sequence.

Example 1 The series associated with the sequence 2, 4, 6, 8, 10 is $2 + 4 + 6 + 8 + 10$.

Example 2 The series associated with the sequence -1, $\frac{1}{2}$, $-\frac{1}{4}$, $\frac{1}{8}$

is $(-1) + (\frac{1}{2}) + (-\frac{1}{4}) + (\frac{1}{8})$.

Example 3 The series associated with 1, 3, 5, 7, 9, …

is $1 + 3 + 5 + 7 + 9 + \ldots + (2n-1) + \ldots$

Examples 1 and 2 are **finite series** while Example 3 is an **infinite series**. A Finite series is associated with a finite sequence in the same way that an infinite series is associated with an infinite sequence.

An infinite series is not a number, but rather an expression which may or may not "sum up" to a number.

Learning Activity 12.1.2.1

20 minutes

1. Write the correct term or terms on the spaces provided.

   a. A collection of numbers arranged in a particular order is called __________.

   b. The indicated sum of the terms of a sequence is called the __________associated with the sequence.

   c. When a sequence has a last term it has a(n) __________ sequence.

   d. The n^{th} term of a sequence is called the __________ term of the sequence.

   e. The numbers in a sequence are called __________ of the sequence.

2. Which of the following sequences below is finite and which is infinite?

   a. 4, 8 12, 16, 20

   b. -1, 1, -1, 1, -1, 1, . . .

   c. 1, 100, 10 000, 1 000 000

   d. $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$

3. In each of the following, the n^{th} term of a sequence is given. Find the first four terms.

   a. $a_n = 3n$

   b. $x_n = n^2 - 2$

   c. $n^2 - 1$

   d. $\frac{n}{n-1}$

12.1. 2.2: Arithmetic Sequences or Progression

In a sequence 2, 5, 8, 11, 14, . . . each term (after the first) is obtained by adding 3 to the term immediately preceding it. That is,

the second term = first term + 3 $\quad 5 = 2 + 3$

the third term = second term + 3 $\quad 8 = 5 + 3$

and so forth. A sequence like this is given a special name called **arithmetic progression.**

> **An arithmetic progression** is a sequence in which every term after the first is the sum of the preceding term and a fixed number called the **common difference** (**d**) of the progression. An arithmetic progression is abbreviated **AP**.

We use the following notation:

$\mathbf{a_1}$	for the first term
$\mathbf{a_n}$	for the $n^{th\ term}$
d	for the common ratio
n	for the number of terms from a_1 to $a_{n,}$ inclusive
$\mathbf{S_n}$	for the sum of the first n terms

Identifying Arithmetic Sequence or Progression

Example 1

Determine if each sequence could be arithmetic. If so, give the common difference.

a. 8, 13, 18, 23, 28,...

Solution: Find the difference of each and the term before it.

8 13 18 23 28, . . .

5 5 5 5

Therefore, the sequence could be arithmetic with common difference of 5.

b. 1, 2, 4, 8, 16, . . .

Solution: Find the difference of each and the term before it.

1 2 4 8 16, . . .

1 2 4 8

Therefore, the sequence is not arithmetic.

c. 5, 1, -3, -7, -11, . . .

Solution: Find the difference of each and the term before it.

$$\begin{array}{ccccccccc} 5 & & 1 & & -3 & & -7 & & -11, \ldots \\ & -4 & & -4 & & -4 & & -4 & \end{array}$$

Therefore, the sequence could be arithmetic with common difference of -4.

d. $1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \ldots$

Solution: Find the difference of each and the term before it.

$$\begin{array}{ccccccccccccc} 1 & & \frac{3}{2} & & 2 & & \frac{5}{2} & & 3 & & \frac{7}{2} & & 4, \ldots \\ & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & \end{array}$$

Therefore, the sequence could be arithmetic with common difference of $\frac{1}{2}$.

Suppose we want to find the 100th term of the sequence 5, 7, 9, 11, 13, ...

If you do not want to find the 99 terms, you could look for a pattern in the terms of the sequence.

Term Name	a_1	a_2	a_3	a_4	a_5	a_6
Term	5	7	9	11	13	15
Pattern	5 + 0(2)	5 + 1(2)	5 + 2(2)	5 + 3(2)	5 + 4(2)	5 + 5(2)

The common difference (d) is 2. For the 2nd term, one 2 is added to a_1. For the third term, two 2's are added to a_1. The pattern shows that for each term, the number of 2's added is one less than the **term number**, or (n – 1). The 100th term is the first term, 5, plus 99 times the common difference, 2.

$$\begin{aligned} a_{100} &= 5 + 99(2) \\ &= 5 + 98 \\ &= 203 \end{aligned}$$

Hence, the formula used to find the nth term (a_n) of arithmetic sequence with common difference (d) is

$$\boxed{a_n = a_1 + (n-1)d}$$

Example 1

Write the first five terms of an infinite arithmetic progression having first term $\mathbf{a_1}$ = 3 and common difference $\mathbf{d}$ = 2.

Solution: $a_1 = 3$

$a_2 = a_1 + (n - 1)d = 3 + (2 - 1)(2) = 3 + 2 = 5$

$a_3 = a_1 + (n - 1)d = 3 + (3 - 1)(2) = 3 + 4 = 7$

$a_4 = a_1 + (n - 1)d = 3 + (4 - 1)(2) = 3 + 6 = 9$

$a_5 = a_1 + (n - 1)d = 3 + (5 - 1)(2) = 3 + 8 = 11$

Therefore, the AP is 3, 5, 7, 9, 11, …

Example 2

Find the 12^{th} term of the arithmetic sequence -9, -5, -1, 3, …

Solution: $a_1 = -9$, $d = 4$, $n = 12$

Substitute these in the formula: $a_n = a_1 + (n - 1)d$

We have $a_{12} = -9 + (12 - 1)(4)$

$= -9 + (11)(4)$

$= -9 + 44$

$\mathbf{= 35}$

Example 3

Find the 23^{rd} term of the arithmetic progression 25, 21, 17, 13, …

Solution: $a_1 = 25$, $d = -4$, $n = 23$

Substitute these in the formula: $a_n = a_1 + (n - 1)d$

We have $a_{23} = 25 + (23 - 1)(-4)$

$= 25 + (22)-4$

$= 25 - 38$

$= -63$

You can use the formula for the n^{th} term of an arithmetic progression to solve for other variables.

Example 4

Find the 5^{th} term and the 11^{th} term of the arithmetic progression with first term 3 and common difference 4.

Solution: We are given that $\mathbf{a_1} = 3$ and $\mathbf{d} = 4$

Substitute these to the formula: $a_n = a_1 + (n - 1)d$

Hence, $\mathbf{a_5} = a_1 + (\mathbf{5} - 1)d = 3 + (4)(4) = \mathbf{19} \longleftarrow 9^{th}$ term

$\mathbf{a_{11}} = a_1 + (\mathbf{11} - 1)d = 3 + (10)(4) = \mathbf{43} \longleftarrow 11^{th}$ term

Notice that the **n** in $\mathbf{a_n}$ is the same as the **n** in $(\mathbf{n - 1})$.

As you learnt earlier, associated with a sequence is a series. So, associated with an arithmetic sequence or progression is an arithmetic series.

An **arithmetic series** is the sum of the terms in an arithmetic progression.

We can also find and calculate the sum of the first **n** terms in an arithmetic progression by using the formula

$$\boxed{\mathbf{S_n = \frac{n}{2}[2a_1 + (n - 1)d]}}$$

Example 5

Find the 9^{th} term and the sum of the first 9 terms of the arithmetic progression with the first term ($\mathbf{a_1}$) = -2 and a common difference ($\mathbf{d}$) = 5.

Solution: We must calculate 9^{th} term ($\mathbf{a_9}$) and $\mathbf{S_9}$.

i) Calculate $\mathbf{a_9}$ using the formula $\mathbf{a_n = a_1 + (n - 1)d}$, where: $n = 9$, $a_1 = -2$, $d = 5$

Hence we have

$$\begin{aligned} \mathbf{a_9} &= a_1 + (9 - 1)d \\ &= -2 + (9 - 1)5 \\ &= -2 + 8(5) \\ &= -2 + 40 \\ &= \mathbf{38} \longleftarrow 9^{th} \text{ term} \end{aligned}$$

ii) Calculate S_9 using the formula $S_n = \frac{\mathbf{n}}{\mathbf{2}}[2a_1 + (n-1)d]$, where: n = 9, a_1 = -2, d = 5

Hence we have

$$S_9 = \frac{9}{2}[2(-2)+(9-1)5]$$

$$= \frac{9}{2}[-4+(8)(5)]$$

$$= \frac{9}{2}[-4+40]$$

$$= \frac{9}{2}[36]$$

$$\mathbf{= 162} \longleftarrow \text{sum of the first 9}$$

When we know the **n**$^{\text{th}}$ term it is easier to find the sum of the first n terms of the arithmetic progression using the second formula

$$\boxed{\mathbf{S_n = \frac{n}{2}[a_1 + a_n] \quad \text{or } S_n = n\left(\frac{a - a_n}{2}\right)}}$$

Where: $\mathbf{n \neq a_n}$

For example, in the preceding example, we had calculated $\mathbf{a_9}$ = 38. When we know the n$^{\text{th}}$ term, it is easier to substitute in the formula.

$$S_n = \frac{\mathbf{n}}{\mathbf{2}}[a_1 + a_n] = \frac{9}{2}[-2+38] = \frac{9}{2}[36] = \mathbf{162}$$

Caution: Do not substitute n (which is 9) for a_n (which is 38). That is $\mathbf{n \neq a_n}$.

Example 4

Find the 20$^{\text{th}}$ term and the sum of the first 20 terms of the arithmetic progression below:

-7, -4, -1, 2, ...

Solution: We have a_1 = -7.

i) To find the common difference (d) we take any term and subtract the one preceding it.
In this case, d = -4 – (-7) = -4 + 7 = **3**

ii) Calculate a_{20} using the formula $a_n = a_1 + (n-1)d$, where: n = 20, a_1 = -7, d = 3

Hence we have $A_{20} = -7 + (20-1)3$

$$= -7 + (19)3$$

$$= -7 + 57$$

$$\mathbf{= 50} \longleftarrow 20^{\text{th}} \text{ term}$$

iii) Calculate S_9 using the formula $\mathbf{S_n = \frac{n}{2}[a_1 + a_n]}$, where n = 20, a_1 = -7, a_n = 50

Hence we have $S_9 = \frac{n}{2}\left[a_1 + a_n\right]$

$= \frac{20}{2}\left[-7 + 50\right]$

$= 10\left[43\right]$

$\mathbf{= 430}$ ← sum of the first 20 terms

Example 5

If a clock strikes only at the hours, how many times does it strike from 1 p.m. through midnight?

Solution: We have $a_1 = 1$; $a_n = 12$; $n = 12$

Calculate S_9 using the formula $\mathbf{S_n = \frac{n}{2}[a_1 + a_n]}$, where n = 12, a_1 = -7, a_n = 12

Hence we have $S_9 = \frac{n}{2}\left[a_1 + a_n\right]$

$= \frac{12}{2}\left[1 + 12\right]$

$= 6\left[13\right]$

$\mathbf{= 78}$

Therefore, the number of times the clock strikes is 78.

Example 6

A theatre has 50 rows with 20 seats in the first row, 22 in the second row, 24 in the third row and so forth. How many seats are in the theatre?

Solution: We have $a_1 = 20$; $d = 2$; and $n = 50$, and we must calculate S_{50}.

To calculate S_{50}, use the formula: $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$S_{50} = \frac{50}{2}[2(20 + (50-1)2]$

$= 25[2(20) + 49(2)]$

$= 25[40 + 98]$

$= 25[138]$

$\mathbf{= 3450}$

Therefore, the theatre has 3450 seats.

Example 7

A rock dislodged by a mountain climber falls approximately 16 feet during the first second, 48 feet during the second second, 80 feet during the third second and so on. Find the distance it falls during the tenth second, and the total distance it falls during the first 12 seconds.

Solution: Given: $a_1 = 16$ $a_2 = 48$ $a_3 = 80$

First we find the common difference (d) = 48 – 16 = 32

a. Distance during the 10^{th} second: $d = 32$; $n = 10$; $a_1 = 16$

Substitute these in the formula: $a_n = a_1 + (n - 1)d$

We have

$$
\begin{aligned}
a_{10} &= 16 + (10 - 1)(32) \\
&= 16 + 9(32) \\
&= 16 + 288 \\
&= 304
\end{aligned}
$$

Therefore, distance during the 10^{th} second is 304 feet.

Distance during the 12^{th} second: $d = 32$; $n = 12$; $a_1 = 16$

Substitute these in the formula: $a_n = a_1 + (n - 1)d$

We have

$$
\begin{aligned}
a_{12} &= 16 + (12 - 1)(32) \\
&= 16 + 11(32) \\
&= 16 + 352 \\
&= 368
\end{aligned}
$$

Therefore, distance during the 12^{th} second is 3368 feet.

b. Total distance rock falls during the first 12 seconds

$$S_n = \frac{n}{2}[a_1 + a_n]$$

$$
\begin{aligned}
S_{12} &= \frac{12}{2}(16 + 368) \\
&= 6(384) \\
&= 2304
\end{aligned}
$$

Therefore, distance during the first 12^{th} second is 3363 feet.

Now do the learning activity on the next page.

Learning Activity 12.1.2.2

20 minutes

1. Write the correct term or terms on the spaces provided.

   i. In an arithmetic progression, each term after the first is obtained by ______________ a fixed number called the ______________ to the preceding term.

   ii. The formula for calculating the n^{th} term of an arithmetic progression is ______________.

2. Are the following arithmetic progressions? Answer Yes or No. If YES, give the common difference.

   i. 3, 6, 9, 12, ...

   ii. 12, 5, -2, -9, -16, ...

   iii. 2, 4, 6, 8,...

   iv. 5, 8, 11, 14,...

   v. 4, 9, 14, 19,...

3. With the given information about arithmetic progression, find the indicated unknown.

   i. Given: $a_1 = 6$; $d = 3$

      Find: a_5; S_5

ii. Given: $a_2 = 3$; $a_4 = 7$
Find: d; a_1

iii. Given: $a_2 = 3$; $a_4 = 5$
Find: d; a_1

4. Find the sum of the first 50 odd numbers.

5. How many terms of the arithmetic sequences 5, 7, 9, ... must be added to get 572?

6. If a clock strikes only at the hours, how many times does it strike in a day?

7. An auditorium has 40 rows with 30 seats in the first row, 33 in the second row, 36 in the third row and so forth. How many seats are in the auditorium?

12.1 2.3: Geometric Sequences and Series

The next special type of sequence you will study is geometric progression. In the following sequence each term after the first can be found by multiplying the preceding term by 3.

1, 3, 9, 27, 81,...

> **A geometric progression** is a sequence for which every term after the first is the product of the preceding term and a fixed number called the **common ratio** of the progression.

We use the following notation:

a_1	for the first term
a_n	for the $n^{th\ term}$
r	for the common ratio
n	for the number of terms from a_1 to a_n, inclusive
S_n	for the sum of the first n terms

For example, for the geometric progression $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

$a_1 = 1$ and $r = \frac{1}{2}$.

To obtain each succeeding term of the progression, multiply the preceding 1 by the common difference, $\frac{1}{2}$.

Thus,

$$a_2 = a_1\left(\frac{1}{2}\right) = 1\left(\frac{1}{2}\right) = \frac{1}{2},$$

$$a_3 = a_2\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4},$$

$$a_4 = a_3\left(\frac{1}{2}\right) = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{8},$$

$$a_5 = a_4\left(\frac{1}{2}\right) = \left(\frac{1}{8}\right)\left(\frac{1}{2}\right) = \frac{1}{16},$$

and so forth.

Notice that if we divide any term by the preceding one, the quotient is always $\frac{1}{2}$.

Examples $\frac{1}{2} \div 1 = \frac{1}{2}$; $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$; $\frac{1}{8} \div \frac{1}{4} = \frac{1}{2}$; $\frac{1}{16} \div \frac{1}{8} = \frac{1}{2}$; etc.

There is also a formula for calculating the n^{th} **term** or (general term) of a geometric progression, as with arithmetic progression.

Let us calculate several terms of an arbitrary geometric progression.

1^{st} term $= a_1 = a_1r^0$

2^{nd} term $= a_2 = a_2l = a_2r^1$

3^{rd} term $= a_3 = a_3(r^2) = a_3r^2$

4^{th} term $= a_4 = a_4(r^3) = a_4r^3$

5^{th} term $= a_5 = a_5(r^4) = a_5r^4$

Notice that in each case, the n^{th} term (5^{th} for example) is the first term times r raised to the $(n-1)$ power $(5-1=4)$. Thus, we have

$$\boxed{\mathbf{n^{th}\ term = a_n = a_1r^{(n-1)}}}$$

Note: $\mathbf{a_1}$ is often referred to as **a**.

Example 1

Find the 7^{th} and 10^{th} terms of the geometric progression 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, ...

Solution: We are given: 1^{st} term $(a_1) = 9$ and common ratio $l = \frac{1}{3}$

Hence, using the formula: $\mathbf{a_n = a_1r^{(n-1)}}$, find the 7^{th} and 10^{th} terms.

i. 7^{th} term $= a_7 = a_1r^{(n-1)}$

$= 9(\frac{1}{3})^{7-1}$

$= 9(\frac{1}{3})^6$

$= 9(\frac{1}{3^6})$

$= 9(\frac{1}{729})$

$= \frac{1}{81}$

ii. 10^{th} term $= a_{10} = a_1r^{(n-1)}$

$= 9(\frac{1}{3})^{10-1}$

$= 9(\frac{1}{3})^9$

$= 9(\frac{1}{3^9})$

$= 9(\frac{1}{19{,}683})$

$= \frac{1}{2137}$

Notice that the **n** in a_n is the same as the **n** in the expression $(\mathbf{n-1})$.

Example 2

Find the ninth term of the geometric progression 2, 6, 18, …

Solution: We are given the 1st term (a_1) = 2 and common ratio r = 3

$$a_9 = a_1r^{(n-1)}$$
$$= 2(3)^{9-1}$$
$$= 2(3)^8$$
$$= 2(6561)$$
$$= 13\ 122$$

Example 3

Find the 8th term of the geometric progression 5, 15, 45, … .

Solution: We are given the 1st term (a_1) = 5 and common ratio r = 3

$$a_9 = a_1r^{(n-1)}$$
$$= 5(3)^{8-1}$$
$$= 5(3)^7$$
$$= 5(2187)$$
$$= 10\ 935$$

Below is the formula for calculating the sum of the first n terms of a geometric progression.

$$\boxed{S_n = \frac{a_1 - a_1r^n}{1-r}}$$

We can also derive an alternate formula for $\mathbf{S_n}$ by writing $a_1r^n = r(a_1r^{n-1}) = ra_n$ from the formula $S_n = \frac{a_1 - a_1r^n}{1-r}$.

Hence, the formula becomes $\boxed{S_n = \frac{a_1 - ra_n}{1-r}}$

where:

a_1	is the 1st n term
a_n	is the nth term
r	is the common ratio
S_n	is the sum of the first n terms

Example 4

Find the 8[th] term and the sum of the first 8 terms of the geometric progression below.

$$-2, 1, -\frac{1}{2}, \frac{1}{4}, \ldots$$

Solution: We are given $a_1 = -2$ and $r = -\frac{1}{2}$. Since r is negative, the signs of the terms alternate.

i) To find the 8[th] term, use the formula: $a_n = a_1r^{(8-1)}$

$$a_8 = -2(-\frac{1}{2})^7$$

$$a_8 = -2(-\frac{1}{2^7})$$

$$= -2(-\frac{1}{128})$$

$$= \mathbf{\frac{1}{64}}$$

ii) To find the sum of the first 8 terms, use the formula: $S_n = \frac{\mathbf{a_1 - a_1r^n}}{\mathbf{1 - r}}$.

$$S_8 = \frac{\mathbf{a_1 - a_1r^n}}{\mathbf{1-r}} = \frac{-2-(-2)(-\frac{1}{2})^8}{1-(-\frac{1}{2})} = \frac{-2-(-2)(-\frac{1}{2})(-\frac{1}{2})7}{\frac{3}{2}}$$

$$= \frac{-2-(-\frac{1}{2})^7}{\frac{3}{2}}$$

$$= \frac{-2-(-\frac{1}{128})}{\frac{3}{2}}$$

$$= \frac{-2+\frac{1}{128}}{\frac{3}{2}}$$

$$= \frac{-256+1}{128}\left(\frac{2}{3}\right)$$

$$= \frac{-255}{128}\left(\frac{2}{3}\right)$$

$$= -\frac{\mathbf{85}}{\mathbf{64}}$$

We can also calculate the sum of the first 8 terms using the alternate formula, $\mathbf{S_n = \frac{a_1 - ra_n}{1 - r}}$, which might have been easier as shown.

$$S_n = \frac{a_1 - ra_n}{1-r} = \frac{-2-(-\frac{1}{2})(\frac{1}{64})}{1-(-\frac{1}{2})}$$

$$= \frac{-2+\frac{1}{128}}{\frac{3}{2}}$$

$$= \frac{-256+1}{128}\left(\frac{2}{3}\right)$$

$$= \frac{-255}{128}\left(\frac{2}{3}\right)$$

$$= -\mathbf{\frac{85}{64}}$$

Example 5

Find the 10^{th} term and the sum of the first 10 terms of the following geometric progression:

1, -2, 4, -8,...

Solution: In this problem, we have $a_1 = 1$ and $r = -2$

Hence, i) the 10^{th} term = $a_{10} = a_1 r^{10-1}$

$$= (1)(-2)^9$$

$$= -2^9$$

$$= \mathbf{-512}$$

Since we calculated $a_{10} = -512$, it is easier to calculate the S_{10} by substituting it with the formula for the sum.

Hence, ii) The sum of the first 10 terms = $S_{10} = \frac{a_1 - ra_{10}}{1 - r}$

$$= \frac{1-(-2)(-512)}{1-(-2)}$$

$$= \frac{1-1024}{3}$$

$$= -\frac{1023}{3}$$

$$= \mathbf{-341}$$

Example 6

Find a_1 and r for a geometric progression that has $a_2 = 10$ and $a_5 = 80$.

Solution: $80 = a_5 = a_1r^{5-1} = a_1r^4$
$10 = a_2 = a_1r^{2-1} = a_1r$

Divide these terms: $\frac{80}{10} = \frac{a_5}{a_2} = \frac{a_1r^4}{a_1r} = r^3$

$8 = r^3$

$2 = r$

Then $10 = a_2 = a_1r = a_12$

$10 = a_12$

$5 = a_1$

Thus, $\mathbf{a_1 = 5}$ **and** $\mathbf{r = 2}$**.**

The next examples are problems in **practical situation involving geometric progression.**

When solving real life problems involving geometric progression, it is wise to write out the first few terms. Once this is done, the first term (**a_1**) and the common ratio (**r**) are much easier to identify.

Example 7

A second hand car costing K8000 depreciates 20% of its value each year. How much is the car worth at the end of 5 years?

Solution: At the beginning of the 1st year (**a_1**) = K8000
At the beginning of the 2nd year (a_2) = (0.80)(K8000) = K6400
At the beginning of the 3rd year (a_3) = $(0.80)^2$(K8000) = (0.80)(K6400) = K5120
At the beginning of the 4th year (a_4) = $(0.80)^3$(K8000) = (0.80)(K5120) = K4096

Notice that to obtain the next term in the sequence, the preceding term is multiplied by 0.80 (80%) giving the value of the car each succeeding year.

Thus a_1 = K8000 and r = 0.80 or 0.8.

Now we must find a_6 to calculate the value of the car at the end of 5 years. (Note that the value of the car at the end of 5th year is equal to the value at the beginning of the 6th year).

Thus, n = 6.

Hence, $a_6 = a_1r^{6-1} = (8000)(0.80)^5$

$= 2621$

Therefore the value of the car after five years is K2126.

Example 8

Sarah borrows K1000 at 12% interest compounded annually. If she pays off the loan in full at the end of 3 years, how much will she pay?

Solution: At the beginning of 1^{st} year (a_1) = 1000

At the beginning of 2^{nd} year or end of 1^{st} year (a_2)

$$a_2 = 1000 + 0.12(1000)$$
$$= (1 + 0.12(1000)$$
$$= (1.12)(1000)$$

At the beginning of 3^{rd} year or end of 2^{nd} year (a_3)

$$a_3 = (1.12)(1000) + (0.12)(1.12)(1000)$$
$$= (1 + 0.12)(1.12)(1000)$$
$$= (1.12)^2(1000)$$

To obtain the next term in the sequence, the preceding term is multiplied by 1.12.

To find the amount that would have to be repaid at the end of 3 years, we calculate $\mathbf{a_4}$.

Thus n = 4.

We have: $a_1 = 1000$ and $r = 1.12$

Hence, $a_4 = a_1r^{4-1}$

$$= (1000)(1.12)^3$$
$$= (1000)(1.40493)$$
$$= 1404.93$$

Therefore, at the end of 3 years, Sarah would have to pay back K1404.93.

Example 9

A rubber ball is dropped from a height of 18 metres. Each time it strikes the ground, it rebounds $\frac{2}{3}$ of the distance through which it fell.

(a) Through what distance did the ball fall when it struck the ground for the 5^{th} time?

(b) Through what distance has it travelled from the time it was dropped until it struck the ground for the fifth time?

Solution:

(a) Let a_1 be the distance the ball fell when it struck the ground the 5^{th} time

Hence, $a_1 = 18$ m; $n = 5$; $r = \frac{2}{3}$

Substituting these to the formula $a_n = ar^{n-1}$, we obtain

$$a_5 = 18(\frac{2}{3})^{5-1}$$

$$= 18(\frac{2}{3})^4$$

$$= 18(\frac{16}{81})$$

$$= \frac{288}{81}$$

$$= \frac{32}{9} \textbf{ m or 4 m (to the nearest metre)}$$

(b) Let S_5 be the distance travelled by the ball in 5 successive falls and S_4 be the distance of the ball in 4 successive rebounds.

Then, the total distance travelled by the ball for the 5 falls and first 4 rebounds is given by $S_5 + S_4$.

For the falls, we have: $a_1 = 18$ m; $n = 5$; $r = \frac{2}{3}$

Substitute these data with formula $S_n = \frac{a_1 - a_1r^n}{1 - r}$ to find the distance for the falls.

Hence, $S_5 = \dfrac{18 - 18\left(\frac{2}{3}\right)^5}{1 - \frac{2}{3}}$

$$= \frac{18 - 18\left(\frac{2}{3}\right)^5}{\frac{1}{3}}$$

$$= \frac{18 - 18\left(\frac{32}{243}\right)}{\frac{1}{3}}$$

$$= \frac{18 - \left(\frac{576}{243}\right)}{\frac{1}{3}}$$

$$= 18 - \frac{64}{27}\left(\frac{3}{1}\right)$$

$$= \frac{486 - 64}{27}(3)$$

$$= \frac{422}{9}\text{ m}$$

For the rebounds: $a_1 = 12;$ $n = 4;$ $r = \frac{2}{3}$

$$S_4 = \frac{12 - 12\left(\frac{2}{3}\right)^4}{1 - \frac{2}{3}}$$

$$= \frac{12 - 12\left(\frac{2}{3}\right)^4}{\frac{1}{3}}$$

$$= \frac{12 - 12\left(\frac{16}{81}\right)}{\frac{1}{3}}$$

$$= \frac{12 - \left(\frac{192}{81}\right)}{\frac{1}{3}}$$

$$= 12 - \frac{64}{27}\left(\frac{3}{1}\right)$$

$$= \frac{324 - 64}{27}(3)$$

$$= \frac{260}{9}\text{ m}$$

Therefore, $S_5 + S_4 = \frac{422}{9}\text{ m} + \frac{260}{9}\text{ m} = \mathbf{\frac{682}{9}\text{ m}}$

Infinite Geometric Series

An infinite **geometric series** is the sum of an **infinite geometric sequence**. This series would have no last term.

The general form of the infinite geometric series is

$$\mathbf{a_1 + a_1r + a_1r^2 + a_1r^3 + ... + a_1r^n + ...}$$

where $\mathbf{a_1}$ is the first term and **r** is the common ratio.

If r < 1, then the sum of an infinite geometric series is given by the formula:

$$\mathbf{S = \left(\frac{a}{1-r}\right)}$$

as **n** is infinite

Example 1

Find the sum of the infinite geometric series:

$$2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + ... + \frac{2}{5^n}, + ...$$

Solution: We use the formula for the sum of an infinite geometric series.

In this case, a = 2; r = $\frac{1}{5}$.

Thus the sum is: $S = \left(\frac{a}{1-r}\right)$

$$= \frac{2}{1-\frac{1}{5}}$$

$$= \frac{5}{2}$$

Now do the learning activity on the next page.

Learning Activity 12.1.2.3

20 minutes

1. Write the correct term or terms on the spaces provided.

   i. In a geometric progression, each term after the first is obtained by ___________ a fixed number called the ____________ to the preceding term.

   ii. The formula for calculating the n^{th} term of a geometric progression is ___________.

   iii. The general form of an infinite geometric series is an expression of the form ____________.

2. Are the following arithmetic progressions? If Yes, give the common difference.

   i. $4, 2, 1, \frac{1}{2}, \ldots$

   ii. 5, 15, 45, 135,...

3. Write the first 5 terms of the geometric progression with $a_1 = 3$ and $r = 2$.

   Answer:___________

4. The first term of a geometric progression is 4, and the sixth term is 128. Find the common ratio.

   Answer:___________

5. Jane borrows K1000 at 12% interest compounded annually. If she pays the loan in full at the end of 4 years, how much does she pay?

   Answer:___________

12.1. 2.4: Summation Notation

Summation or sigma notation is a concise and convenient way to represent long sums. The symbol used to indicate this operation of adding or summing up a group of numbers is a capital Greek letter Sigma **∑**.

However, the instruction "to take the sum of" is rather unclear without indication of what it is that is to be summed. It is necessary to have a system of notation to specify clearly which values are to be summed.

For example, we often wish to sum a number of terms such as

$$1 + 2 + 3 + 4 + 5$$

where there is an obvious pattern to the numbers involved. The sequence of numbers we have in the example is the sum of the first five whole numbers. Generally, we let X be a general symbol for any one of these sequence of numbers. The set of numbers we have are five X's which are 1, 2, 3, 4, and 5. If we now assign a subscript to each of the X's, we can assign an X a given subscript to each number thus

$$X_1 = 1; \quad X_2 = 2; \quad X_3 = 3; \quad X_4 = 4, \quad X_5 = 5$$

In the case of five numbers, the subscripts will naturally run from 1 to 5, but there are usually more than five numbers involved, and so it is desirable to have a generalized notation, so that we can apply the system to any group of numbers. The general symbol for numbers is **n**.

Any collection of terms or number will consist of **n** terms. If we want a general reference to a single number or term without specifying exactly which, we can use the subscript **r.** Thus X is the r^{th} term.

Consider $X_1 = 1; \quad X_2 = 2; \quad X_3 = 3; \quad X_4 = 4, \quad X_5 = 5$

What is the value of X when r = 5?

The answer is 5. There are five numbers, therefore n = 5.

X, when r = n, must be X_5, which is the symbol for the fifth number, which is 5. In similar fashion, the value of X_1 when r = 2 is 2; when r = 1 it is 1 and so on.

These symbols are often used in connection with the summation sign to indicate exactly which scores are to be summed.

Example: $\sum_{r=1}^{n} X_r$ means the sum of all numbers from X_1 to X_n

The symbols above and below the summation sign are called the **limits of** the summation.

The value of **r** under the summation sign tells you where to start the addition, and the values above the summation sign tells you where to stop. The starting and the stopping places can be anywhere in the set of numbers.

Hence, the sum $X_1 + X_2 + X_3 + X_4 + X_5$ can be written in summation notation as $\sum_{r=1}^{5} X_1$ which means "the sum of all whole numbers from r = 1 and X_5".

The numbers 1 and 5 are called the lower and upper limits of the summation.

Evaluating the summation notation, we have: $\sum_{r=1}^{5} X_1 = 1 + 2 + 3 + 4 + 5$

$=15$

In general the symbol $\sum$ means that we replace r wherever it appears after the summation symbol by 1, then by 2, and so on up to n, and then add the terms up. So, you can write

$$\sum_{r=1}^{4} X_r^{\,2} = X_1^{\,2} + X_2^{\,2} + X_3^{\,2} + X_4^{\,2}$$

$$\sum_{i=2}^{5} XY = X_2Y_2 + X_3Y_3 + X_4Y_4 + X_5Y_5$$

The subscript may be any letter. $\sum_{i=1}$ usually the letters **i, r,** and **k** are used.

Now study the following worked examples.

Example 1

Evaluate: $\sum_{r=1}^{4} r^3$

Solution: This is the sum of all the r^3 terms from r = 1 to r = 4.

So we take each value of r, work out r^3 in each case, and add the results.

Therefore, $\sum_{r=1}^{4} N^3 = 1^3 + 2^3 + 3^3 + 4^3$

$= 1 + 8 + 27 + 64$

$= 100$

Example 2

Evaluate: $\sum_{r=2}^{5} N^2$

Solution: $\sum_{r=2}^{5} N^2 = 2^2 + 3^2 + 4^2 + 5^2$

$= 4 + 9 + 16 + 25$

$\mathbf{= 54}$

Example 3

Evaluate: $\sum_{k=0}^{6} 2^k$

Solution: In this example, you will notice that there are 7 terms in the sum, because we have k = 0.

$\sum_{k=0}^{6} 2^k = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$

$= 1 + 2 + 4 + 8 + 16 + 32 + 64$

$\mathbf{= 127}$

Example 4

Evaluate: $\sum_{k=1}^{5} (4k - 5)$

Solution: In this example, you will notice that there are 5 terms because we have k = 1.

$\sum_{k=1}^{5} (4k - 5) = [4(1) - 5] + [4(2) - 5] + [4(3) - 5] + [4(4) - 5] + [4(5) - 5]$

$= -1 + 3 + 7 + 11 + 15$

$= 35$

Example 5 Evaluate: $\sum_{r=1}^{6} \frac{1}{2} r(r+1)$

Solution: You might notice and recognize that each number $\frac{1}{2}r(r+1)$ is a triangular number, and so this example asks for the sum of the first 6 triangular numbers.

$$\sum_{r=1}^{6} \frac{1}{2} r(r+1) = \frac{1}{2}(1)(2) + \frac{1}{2}(2)(3) + \frac{1}{2}(3)(4) + \frac{1}{2}(4)(5) + \frac{1}{2}(5)(6) + \frac{1}{2}(6)(7)$$

$$= 1 + 3 + 6 + 10 + 15 + 21$$

$$= \mathbf{56}$$

The next examples will show you how to write long sums in sigma or summation notation.

Example 6

Write the sum $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$ using sigma notation.

Solution: We can write

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = \sum_{k=1}^{7} \mathbf{k^3}$$

Example 7

Write the sum $\sqrt{3} + \sqrt{4} + \sqrt{5} + \ldots + \sqrt{77}$ using sigma notation.

Solution: We can write

$$\sqrt{3} + \sqrt{4} + \sqrt{5} + \ldots + \sqrt{77} = \sum_{k=3}^{77} \sqrt{\mathbf{k}}$$

Example 8

Write the sum 2 + 2 + 2 + 2 + 2 + 2 using sigma notation.

Solution: We can write

$$2 + 2 + 2 + 2 + 2 + 2 = \sum_{i=1}^{6} \mathbf{2}$$

Example 9

Write 7 + 14 + 21 + 28 + ... + 105 using sigma notation.

Solution: $7 + 14 + 21 + 28 + \ldots + 105 = \sum_{n=7}^{15} \mathbf{7n}$

Some Rules of Summation

Rule 1 The sum of the sums of two or more variables is equal to the sum of their summation.

$$\boxed{\sum_{i=1}^{n}(x_i + y_i + z_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i}$$

Example 1 Evaluate: $\sum_{i=1}^{4}(2_i + 3_i + 4_i)$

Solution:

$$\sum_{i=1}^{4}(2_i + 3_i + 4_i) = \sum_{i=1}^{n} 2_i + \sum_{i=1}^{n} 3_i + \sum_{i=1}^{n} 4_i$$

$= (2 + 2 + 2 + 2) + (3 + 3 + 3 + 3) + (4 + 4 + 4 + 4)$

$= 8 + 12 + 16$

$= \mathbf{36}$

Rule 2 The sum of a constant and times the values of a variable is equal to the constant times the sum of the variable.

$$\boxed{\sum_{i=1}^{n}(cx_i) = c\sum_{i=1}^{n} x_i \quad \text{where c is constant}}$$

Example 2 Evaluate $\sum_{i=1}^{4}(3x_i)$

Solition: $\sum_{i=1}^{4}(3x_i) = 3\sum_{i=1}^{4} x_i$

$= 3(1 + 2 + 3 + 4)$

$= 3(10)$

$= \mathbf{30}$

Rule 3 The sum of a constant taken n time is the constant times n.

$$\boxed{\sum_{i=1}^{n} c = nc \quad \text{where c is constant}}$$

Example 3 Evaluate: $\sum_{i=1}^{5} 3$

Solution: $\sum_{i=1}^{5} 3 = 5(3) = \mathbf{15}$

Rule 4 The sum of the values of a variable plus a constant is equal to the sum of values of the variable plus n times the constant.

$$\boxed{\sum_{i=1}^{n} (x_i + c) = \sum_{i=1}^{n} x_i + nc}$$

Example 4

Evaluate: $\sum_{i=1}^{4} (x_i + 2)$

Solution: $\sum_{i=1}^{4} (x_i + 2) = (1 + 2) + (2 + 2) + (3 + 2) + (4 + 2)$

$= (1 + 2 + 3 + 4) + 4(2)$

$= 10 + 8$

$= \mathbf{18}$

Example 5

Evaluate: $\sum_{i=1}^{5} \left(x_i^{2} + 2\right)$

Solution: $\sum_{i=1}^{5} \left(x_i^{2} + 2\right) = (1^2 + 2) + (2^2 + 2) + (3^2 + 2) + (4^2 + 2) + (5^2 + 2)$

$= (1^2 + 2^2 + 3^2 + 4^2 + 5^2) + 5(2)$

$= 1 + 4 + 9 + 16 + 25 + 10$

$= \mathbf{65}$

Rule 5 The sum of the values of a variable minus a constant is equal to the sum of values of the variable minus n times the constant.

$$\boxed{\sum_{i=1}^{n}(x_i - c) = \sum_{i=1}^{n} x_i - nc}$$

Example 6 Evaluate: $\sum_{i=1}^{4}(x_i - 2)$

Solution: $\sum_{i=1}^{4}(x_i - 2) = (1 - 2) + (2 - 2) + (3 - 2) + (4 - 2)$

$= (1 + 2 + 3 + 4) - 4(2)$

$= 10 - 8$

$= \mathbf{2}$

Example 6 Evaluate: $\sum_{i=1}^{5}(x_i{}^2 - 3)$

Solution: $\sum_{i=1}^{5}(x_i{}^2 - 3) = (1^2 - 3) + (2^2 - 3) + (3^2 - 3) + (4^2 - 3) + (5^2 - 3)$

$= (1^2 + 2^2 + 3^2 + 4^2 + 5^2) - 5(3)$

$= 1 + 4 + 9 + 16 + 25 - 15$

$= \mathbf{50}$

Learning Activity 12.1.2.4

20 minutes

1. Evaluate each of the following sigma notations.

a. $\sum_{i=1}^{6} 2$

b. $\sum_{k=2}^{5} \frac{1}{k}$

c. $\sum_{i=6}^{10} i$

d. $\sum_{k=1}^{6} 2^{k-1}$

e. $\sum_{n=1}^{5} 2n+3$

2. Write the following in summation or sigma notation.

a. $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2$

b. $3 + 3 + 3 + 3 + 3 + 3 + 3$

3. Write each of the following sums without using sigma notation.

a. $\sum_{k=1}^{5} \frac{1}{k}$

b. $\sum_{k=1}^{6} 2k^1$

c. $\sum_{i=1}^{6} i^2x^i$

NOW DO THE SUMMATIVE TASK 2

SUMMATIVE TASK 12.1.2

30 minutes

A. Choose the correct answer to each question from the alternatives A, B. C and D. Circle your answer.

1. The next term in the sequence 8, 5, 2, -1 is

A. -2 B. -3

C. -4 D. -5

2. Which of the following is an expression for the n^{th} term of a sequence 13, 17, 21, 25,...

A. $12n + 1$ B. $\frac{13n}{2-n}$

C. $4n + 1$ D. $4n + 9$

3. If a and b are the first term and the second term, respectively, of a geometric sequence, the third term is

A. ab B. $\frac{a}{b}$

C. $\frac{b}{a}$ D. $\frac{b^2}{a}$

4. x, x – 2 , x -4 ,... form

A. an arithmetic sequence B. a geometric sequence

C. a finite sequence D. an infinite geometric sequence

5. What are the next three terms of -2, -1, 1, 4, 8, ...

A. 13, 19, 26,... B. 4, 6, 3,...

C. 9, 10, 11,... D. 14, 19, 25,...

B. Find the next three terms in each sequence

1. 1, -3, 9, -27, 81,...

Answer:___________

2. 4, 16, 36, 64, 100,...

Answer:__________

3. $\frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{5}{32}, \ldots$

Answer:__________

4. 0, 3, 8, 15, 24,...

Answer:__________

C. Given the first term and the common difference of an arithmetic progression, find the first five terms of the progression.

1. $a_1 = 28$, $d = 10$

Answer:__________

2. $a_1 = 35$, $d = 4$

Answer:__________

3. $a_1 = -34$, $d = -10$

Answer:__________

D. Determine if the sequence is arithmetic. If it is, find the common difference.

1. 35, 32. 29, 26,...

Answer:__________

2. -3, -23, -43, -63,...

Answer:__________

3. -34, -64, -94, -124,...

Answer:__________

4. 9, 14, 19, 24,...

Answer:__________

5. -7, -9, -11, -13, ...

Answer:__________

E. Determine if the sequence is geometric. If it is, find the common ratio.

1. -3, -15, -75, -375,...

Answer:__________

2. -2, -4, -8, -16,...

Answer:__________

3. -1, 1, 4, 8,...

Answer:__________

4. 1, -5, 25, -125, ...

Answer:__________

5. -1, 6, -36, 216, ...

Answer:__________

F. Given the data on the table

i	x_i
1	-1
2	3
3	7

Evaluate the following:

1. $\sum_{i=1}^{3} x_i$

2. $\sum_{i=1}^{3} x_i^2$

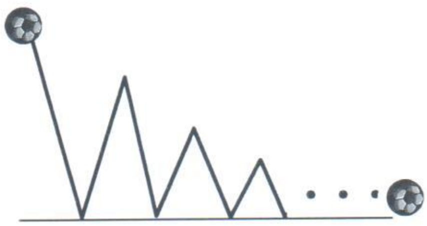
3. $\left(\sum_{i=1}^{3} x_i\right)^2$

G. Problem Solving

1. Zion borrows K1000 at 15% interest compounded annually. If she pays off the loan in full at the end of 3 years, how much does she pay?

Answer:__________

2. A rubber ball is dropped from a height of 9 meters. Each time the ball strikes the ground, it rebounds to a height that is $\frac{2}{3}$ the height from which it last fall. Find the total distance the ball travels before coming to rest?



Answer:__________

3. Mrs. Smith invested a certain amount of money which earned $1\frac{1}{5}$ times as much in the second year as in the first year, and $1\frac{1}{5}$ times as much in the third year as in the second year, and so on. If the investment earned her K22 750 in 3 years, how much would it earn her in the fifth year?

Answer:__________

12.1.3. Binomial Expansion

An expression of the form (a + b) is called a **binomial**. Although in principle it is easy to multiply (a + b) by itself or raise to any power, raising it to a very high power would be tedious or boring.

In this section, we will find a formula that gives the expansion of **$(a + b)^n$** for any natural number **n** and then prove it using mathematical induction.

12.1.3.1 The Pascal's Triangle

Let us first look at the expansion of $(a + b)^n$.

To find a pattern in the expansion of $(a + b)^n$, we first look at some special cases:

$$(a + b)^0 = 1$$

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

From the expansion of $(a + b)^n$, the following simple patterns emerge.

i. There are **n + 1** terms, the first being **a^n** and the last **b^n**.
ii. The exponent of **a** decrease by 1 from term to term while the exponents of **b** increase by 1.
iii. The exponents on **a** start with n and decrease to 0.
iv. The exponents on **b** start with 0 and increase to n.
v. The sum of the exponents in each term is **n**.
vi. If **a** and **b** are both positive, all terms are positive.
vii. If **a** is positive and **b** is negative, the terms have alternating signs: those with odd powers of **b** are negative.
viii. If **a** is negative and **b** is positive, the terms have alternating signs; those with odd powers of a are negative.
ix. If **a** and **b** are both negative, all terms are positive if **n** is even and negative if **n** is odd.

For instance, notice how the exponents of **a** and **b** behave in the expansion of $(a + b)^5$.

The exponents of **a** decrease:

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + b^5$$

The exponents of **b** increase:

$$(a + b)^5 = a^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + b^5$$

With these observations we can write the form of the expansion of $\mathbf{(a + b)^n}$ for any natural number **n**.

For example, writing a question mark for the missing coefficients, we have

$$(a + b)^7 = a^7 + \mathbf{?}a^6b + \mathbf{?}a^5b^2 + \mathbf{?}a^4b^3 + \mathbf{?}a^3b^4 + \mathbf{?}a^2b^5 + \mathbf{?}ab^6 + b^7$$

To complete the expansion we need to determine these coefficients. To find a pattern, let us write the coefficients in the expansion of $(a + b)^n$ for the first few values of **n** in a triangular array as shown in the following diagram.

$$(a + b)^0 = 1$$
$$(a + b)^1 = 1a + 1b$$
$$(a + b)^2 = 1a^2 + 2ab + 1b^2$$
$$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
$$(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$
$$(a + b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

If we omit everything in the above diagrams except the numerical coefficients, we have a triangular array of numbers as shown on the next page.

Row 1						1						Coefficients for n = 0
Row 2					1		1					Coefficients for n = 1
Row 3				1		2		1				Coefficients for n = 2
Row 4			1		3		3		1			Coefficients for n = 3
Row 5		1		4		6		4		1		Coefficients for n = 4
Row 6	1		5		10		10		5		1	Coefficients for n = 5

$6 + 4 = 10$

This triangular array of numbers is known as the **Pascal's Triangle**.

Examining closely the numbers in the Pascal's triangle reveals that:

1. The row number corresponds to the exponent **n** in the expansion of $\mathbf{(a + b)^n}$.
2. The first and the last number in any row are 1.
3. Any other number in Pascal's Triangle is the sum of the two closest numbers in the row above it. (See triangle on the previous page).

Pascal's triangle gives us one way to find the coefficients in a binomial expansion. It can be extended to any size by using the rules given above.

Example 1 Expand $(2x + y)^4$ using the Pascal's triangle.

Solution: In this example **n** = 4, **a** = 2x and **b** = y

Row 6 of the Pascal's triangle has the following numbers for the coefficients.

1 4 6 4 1

Thus, we substitute a = 2x and b = y into

$$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

to obtain

$$(2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4$$

$$= 16x^4 + 4(8x^3)y + 6(4x^2)y^2 + 8xy^3 + y^4$$

$$= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$$

Therefore, $\mathbf{(2x + y)^4 = 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4}$**.**

Example 2

Expand $(m - 3)^5$ using the Pascal's triangle.

Solution: In this example **n** = 5, **a** = m and **b** = -3 (since we are expanding $[n + (-3)]^5$. The signs in the resulting series are alternate since **b** is negative.

Row 5 of the Pascal's triangle has the following numbers for the coefficients.

1 5 10 10 5 1

Thus, we substitute a = m and b = -3 into

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

to obtain

$$= m^5 + 5m^4(-3) + 10m^3(-3)^2 + 10m^2(-3)^3 + 5m(-3)^4 + (-3)^5$$

$$= m^5 - 15m^4 + 10m^3(9) + 10m^2(-27) + 5m(81) - 243$$

$$= m^5 - 15m^4 + 90m^3 - 270m^2 + 405m - 243$$

Therefore, $(m - 3)^5 = m^5 - 15m^4 + 90m^3 - 270m^2 + 405m - 243$.

Example 3

Use the Pascal's triangle to expand $(2 - 3x)^5$.

Solution: In this example n = 5, a = 2 and b = -3x (since we are expanding $[2 + (-3x)]^5$. The signs in the resulting series are alternate since b is negative.

Row 5 of the Pascal's triangle has the following numbers for the coefficients.

1 5 10 10 5 1

Thus, we substitute **a** = 2 and **b** = -3x into

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

to obtain

$$(2)^5 + 5(2)^4(-3x) + 10(2)^3(-3x)^2 + 10(2)^2(-3x)^3 + 5(2)(-3x)^4 + (-3x)^5$$

$$= 32 + 5(16)(-3x) + 10(8)(9x^2) + 10(4)(-27x^3) + 10(81x^4) - 243x^5$$

$$= 32 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243x^5$$

Therefore, $(2 - 3x)^5 = 32 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$.

Now do the learning activity on the next page.

Learning activity 12.1.3.1

20 minutes

1. Expand the Pascal's triangle through row 8.

2. Expand each of the following binomials by using the Pascal's triangle.

   a. $(x + 3)^5$

   b. $(x - 3)^5$

   c. $(x^2 + y)^6$

   d. $(x^{-1} + x)^5$

   e. $(x^2 - 3y)^4$

   f. $(3 - 2)^7$

12.1.3.2: The Binomial Theorem

Although the Pascal's triangle is useful in finding the binomial expansion for reasonably small values of **n**, it is not practical for finding **$(a + b)^n$** for large values of **n**. The reason is that the method we use for finding the successive rows of Pascal's triangle is recursive, (meaning the process is continues and repeated). Thus, to find the 100th row of this triangle, we must first find the preceding 99 rows.

We need to examine the pattern in the coefficients carefully to develop a formula that allows us to calculate directly any coefficient in the binomial expansion.

Such a formula exists. It is commonly called "**n choose k**" because it shows many ways to choose elements from a **set** of **n**.

However, to state the formula we need some notation.

The product of the first **n** natural numbers is denoted by "**n!**" and is called **n factorial**.

$$! = (1)(2)(3)(4).....(n-1)(n)$$

We also define 0! As follows:

$$0! = 1$$

The definition **0!** makes many formulas involving factorials shorter and easier to write.

Examples

a. $4! = (1)(2)(3)(4) = 24$

b. $8! = (1)(2)(3)(4)(5)(6)(7)(8) = 40\ 320$

c. $11! = (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11) = 39\ 916\ 800$

The Binomial Coefficient

So, if we let **n** and **k** be non-negative integers with **$k \leq n$**, the **binomial coefficient** is denoted by $\binom{n}{k}$ and is defined by the formula:

$$\boxed{\binom{n}{k} = \frac{n!}{k!\,(n-k)!}}$$

See examples on the next page.

Example 1 Calculate the binomial coefficients $\binom{9}{4}$.

Solution: $\binom{9}{4} = \frac{9!}{4!(9-4)!}$

$= \frac{\cancel{(1)(2)(3)(4)(5)(6)}(7)(8)(9)}{[(1)(2)(3)(4)][(1)\cancel{(2)(3)(4)(5)}]}$

$= \frac{(6)(7)(8)(9)}{(1)(2)(3)(4)}$

$= \frac{3024}{24}$

= 126

Example 2 Calculate the binomial coefficients $\binom{100}{97}$.

Solution: $\binom{100}{97} = \frac{100!}{97!(100-97)!}$

$= \frac{\cancel{(1)(2)(3)(4)\ldots\ldots(97)}(98)(99)(100)}{\cancel{[(1)(2)(3)(4)\ldots\ldots(97)]}[(1)(2)(3)]}$

$= \frac{(98)(99)(100)}{(1)(2)(3)}$

$= \frac{970\,200}{6}$

= 161 700

Example 3 Calculate the binomial coefficient $\binom{100}{3}$

Solution: $\binom{100}{3} = \frac{100!}{3!(100-3)!}$

$= \frac{\cancel{(1)(2)(3)(\ldots\ldots(97)}(98)(99)(100)}{(1)(2)(3)\cancel{(1)(2)(3)\ldots\ldots(97)}}$

$= \frac{(98)(99)(100)}{(1)(2)(3)}$

$= \frac{970\,200}{6}$

= 161 700

Note: Although the binomial coefficient $\binom{n}{k}$ is defined in terms of fraction, all the results in the two examples are natural numbers. In fact $\binom{n}{k}$ is always a natural number.

Notice that the binomial coefficients in Example 2 and 3 are equal. This is a special case of the following relation:

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

To see any connection between the binomial coefficient and the binomial expansion of $(a + b)^n$, let us calculate the following binomial coefficients:

$$\binom{5}{0} = 1 \quad \binom{5}{1} = 5 \quad \binom{5}{2} = 10 \quad \binom{5}{3} = 10 \quad \binom{5}{4} = 5 \quad \binom{5}{5} = 1$$

$$\binom{5}{0} = \frac{5!}{0!(5-0)!} = \frac{(1)(2)(3)(4)(5)}{(1)(2)(3)(4)(5)} = \frac{20}{20} = 1$$

$$\binom{5}{1} = \frac{5!}{1(5-1)!} = \frac{\cancel{(1)}\cancel{(2)}\cancel{(3)}(4)(5)}{\cancel{(1)}\cancel{(2)}\cancel{(3)}(4)} = \frac{20}{4} = 5$$

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{\cancel{(1)}\cancel{(2)}\cancel{(3)}(4)(5)}{(1)(2)\cancel{(1)}\cancel{(2)}\cancel{(3)}} = \frac{20}{2} = 10$$

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{\cancel{(1)}\cancel{(2)}\cancel{(3)}(4)(5)}{\cancel{(1)}\cancel{(2)}\cancel{(3)}(1)(2)} = \frac{20}{2} = 10$$

$$\binom{5}{4} = \frac{5!}{4!(5-4)!} = \frac{\cancel{(1)}\cancel{(2)}\cancel{(3)}\cancel{(4)}(5)}{\cancel{(1)}\cancel{(2)}\cancel{(3)}\cancel{(4)}(1)} = \frac{5}{1} = 5$$

These precisely are the entries in the fifth row of the Pascal's Triangle. In fact we can write the Pascal's Triangle as follows:

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \cdot \quad \cdot \quad \cdot \quad \binom{n}{n-1} \quad \binom{n}{n}$$

Now we can state the Binomial Theorem.

The Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-1}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^2$$

Let us look at the following example applications of the Pascal's triangle and the Binomial Theorem in expanding binomials.

Example 1

Find the expansion of $(a + b)^7$ using Pascal Triangle.

Solution:

The first term in the expansion is a^7, and the last term is b^7.

Using the fact that the exponent of **a** decreases by 1 from term to term and that of **b** increases by 1 from term to term

$$a^7 + \underline{\ ?\ }a^6b + \underline{\quad}a^5b^2 + \underline{\quad}a^4b^3 + \underline{\quad}a^3b^4 + \underline{\quad}a^2b^5 + \underline{\quad}ab^6 + b^7$$

The appropriate coefficients appear in the seventh row of the Pascal's Triangle. Thus,

$$(a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

Example 2

Use the Pascal's triangle to expand $(2 - 3x)^5$.

Solution:

First we find the expansion of $(a + b)^5$ and then substitute 2 for a and -3x for b.

Using the Pascal's triangle for the coefficients, we get:

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^4b^3 + 5a^5b^4 + b^5$$

Substituting a = 2 and =-3x gives us

$$(2 - 3x)^5 = 2^5 + 5(2)^4(-3x) + 10(2)^3(-3x)^2 + 10(2)^2(-3x)^3 + 5(2)(-3x)^4 + (-3x)^5$$

$$\mathbf{= 32 - 240x^5 + 720x^2 - 1080x^3 + 810x^4 - 243x^5}$$

Example 3

Expand $(x + y)^4$ using the Binomial theorem.

Solution: Substitute **x** for **a** and **y** for **b** in the formula.

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

Verify that $\binom{4}{0} = 1$, $\binom{4}{1} = 4$ $\binom{4}{2} = 8$ $\binom{4}{3} = 4$ $\binom{4}{4} = 1$

So, $\mathbf{(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4}$

Example 4

Use the Binomial Theorem to expand $(\sqrt{x} - 1)^8$.

Solution: We first find the expression $(a + b)^8$ and then substitute $\sqrt{x}$ for a and -1 for b.

Using the Binomial Theorem, we have

$$(a+b)^8 = \binom{8}{0}a^8 + \binom{8}{1}a^7b + \binom{8}{2}a^6b^2 + \binom{8}{3}a^5b^3 + \binom{8}{4}a^4b^4 + \binom{8}{5}a^3b^5 + \binom{8}{6}a^2b^6 + \binom{8}{7}ab^7 + \binom{8}{8} = 1$$

Verify that

$$\binom{8}{0} = 1 \quad \binom{8}{1} = 8 \quad \binom{8}{2} = 28 \quad \binom{8}{3} = 56 \quad \binom{8}{4} = 70 \quad \binom{8}{5} = 56 \quad \binom{8}{6} = 28 \quad \binom{8}{7} = 8 \quad \binom{8}{8} = 1$$

So, $(a + b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$

Performing the substitution **a** = $\sqrt{x}$ or $x^{\frac{1}{2}}$ and **b** = -1 gives

$$(\sqrt{x} - 1)^8 = (x^{\frac{1}{2}})^8 + 8(x^{\frac{1}{2}})^7(-1) + 28(x^{\frac{1}{2}})^6(-1)^2 + 56(x^{\frac{1}{2}})^5(-1)^3 + 70(x^{\frac{1}{2}})^4(-1)^4 + 56(x^{\frac{1}{2}})^3(-1)^5 +$$
$$28(x^{\frac{1}{2}})^2(-1)^6 + 8(x^{\frac{1}{2}})(-1)^7 + (-1)^8$$

This simplifies to $(\sqrt{x} - 1)^8 = x^4 - 8x^{\frac{7}{2}} + 28x^3 - 56x^{\frac{5}{2}} + 70x^2 - 56x^{\frac{3}{2}} + 28x - 8x^{\frac{1}{2}} + 1$

The Binomial Theorem can be used to find a particular term of a binomial expansion without having to find the entire expansion.

The General Term of the Binomial Expansion

The term that contains $\mathbf{a^r}$ in the expression $\mathbf{(a + b)^n}$ is $\mathbf{\binom{n}{n-r} a^r b^{n-r}}$

Example 5

Find the term that contains x^5 in the expansion of $(2x + y)^{20}$

Solution: The term that contains 5 is given by the formula for the general term with $a = 2x$; $b = y$; $n = 20$ and $r = 5$.

So this term is

$$\binom{n}{n-r} a^r b^{n-r} = \binom{20}{20-5} 2x^5 y^{20-5}$$

$$= \binom{20}{15} a^5 b^{15}$$

$$= \frac{20!}{15!(20-15)!}(2x)^5 y^{15}$$

$$= \frac{20!}{15!\,5!} 32x^2 y^{15}$$

$$= \frac{\cancel{(1)(2)(3)\ldots(15)}\ldots(20)}{\cancel{(1)(2)(3)\ldots(15)}(1)(2)(3)(4)(5)} 32x^2 y^{15}$$

$$= \frac{1680480}{120} 32x^2 y^{15}$$

$$= (15\ 004)\ 32x^2 y^{15}$$

$$\mathbf{= 496\ 128\ x^2 y^{15}}$$

Example 6

Find the coefficient of x^8 in the expansion of $\left[x^2 + \frac{1}{x}\right]^{10}$.

Solution: Both x^2 and $\frac{1}{x}$ are powers of x, so the power of x in each term of the expansion is determined by both terms of the binomial. To find the required coefficient, we first find the general term in the expansion.

By the formula we have $a = x^2$ and $b = \frac{1}{x}$ and $n = 10$

So the general term is: $\binom{10}{10-r}(x^2)^r\left(\frac{1}{x}\right)^{10-r}$

$$= \binom{10}{10-r}(x^{2r})(x^{-1})^{10-r}$$

$$= \binom{10}{10-r}x^{3r-10}$$

Thus the term that contains x^8 is the term in which

$$3r - 10 = 8$$

$$r = 6$$

so the required coefficient is $\binom{10}{10-6} = \binom{10}{4}$ = **210**

Learning Activity 12.1.3.2

20 minutes

1. Use the Pascal's Triangle to expand the following expressions.

   a. $(x - 1)^6$

   b. $(x + 2)^3$

2. Consider the expansion $(x + b)^{30}$.

   a. What is the exponent of b in the 1^{st} term?

   Answer: ___________

   b. What is the exponent of b in the 3^{rd} term?

   Answer: ___________

   c. Write the fourth term with its coefficient.

   Answer: ___________

NOW DO THE SUMMATIVE TASK 3

SUMMATIVE TASK 12.1.3

30 minutes

1. Use the Pascal's Triangle to expand the following expressions.

(a) $(x + y)^6$

Answer: ___________

(b) $(2x + 1)^4$

Answer: ___________

(c) $(x + 1)^7$

Answer: ___________

(d) $(2m + 3)^3$

Answer: ___________

(e) $(1 - 2x)^5$

Answer: ___________

2. Expand using the Binomial Theorem.

(a) $(x + y)^6$

Answer: ___________

(b) $(1 + 2y)^5$

Answer: ___________

(c) $(3x - y^2)^5$

Answer: ___________

3. Find the third term of the expression $(x + 3y)^2$.

Answer: ___________

4. Find the last term of the expression $(a - 2b)^4$.

Answer: ___________

5. Find the coefficient of x^5 in the expansion of $(x^2 + \frac{1}{x})^{10}$.

Answer: ___________

12.1.4 Determinants

In this topic, we shall take up determinants and apply these in solving system of linear equations. You have learned other methods of solving system of linear equations in your Grade 11 Mathematics.

12.1.4.1 Basic Concepts

Let us define determinant.

Determinant is a square array of numbers or variables enclosed between vertical lines.

A determinant represents the sum of certain products.

The **elements** of a determinant are the numbers in its array.

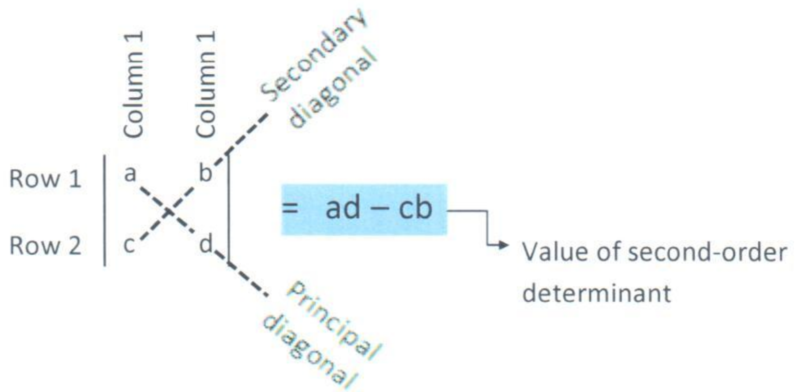
A **row** of a determinant is a horizontal line of its elements.

A **column** of a determinant is a vertical line of its elements.

The **principal diagonal** of a determinant is the line of its elements from the upper left corner to the lower right corner.

The **secondary diagonal** of a determinant is the line of elements from the lower left corner to the upper right corner.

Below is a diagram showing a second-order determinant.



Determinants are similar to absolute value and use the same notation, but they are not identical. Just as absolute values can be evaluated and simplified to get a single number, so can determinant.

One of their differences is that determinants can be negative value.

Evaluating a Determinant of the Second Order

The value of a second order determinant is found by finding the difference between the product of the numbers in the principal diagonal and the product of the numbers in the secondary diagonals.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Examples

Find the value of each second-order determinants.

1. $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = (3)(2) - (1)(4) = 6 - 4 = \mathbf{2}$

2. $\begin{vmatrix} -5 & -6 \\ 2 & 3 \end{vmatrix} = (-5)(3) - (2)(-6) = -15 - (-12) = -15 + 12 = \mathbf{-3}$

3. $\begin{vmatrix} 5 & -7 \\ 4 & 0 \end{vmatrix} = (5)(0) - (4)(-7) = 0 - (-28) = 0 + 28 = \mathbf{28}$

4. $\begin{vmatrix} 5 & 0 \\ -9 & 8 \end{vmatrix} = (5)(8) - (-9)(0) = 40 + 0 = 40$

5. $\begin{vmatrix} 2 & -4 \\ -3 & 6 \end{vmatrix} = (2)(6) - (-3)(-4) = 12 - (12) = 12 - 12 = 0$

6. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 5 & 2 \\ -2 & 6 \end{vmatrix}$

$= [(1)(4) - (3)(2)] + [(5)(6) - (-2)(2)]$

$= (4 - 6) + (30 + 4)$

$= -2 + 34$

$= 32$

7. $\begin{vmatrix} 5 & 2 \\ -2 & 6 \end{vmatrix} - \begin{vmatrix} 6 & -7 \\ 4 & 0 \end{vmatrix}$

$= [(5)(6) - (-2)(2)] - [(6)(0) - (4)(-7)]$

$= [30 + 4] - [0 + 28]$

$= 34 - 28$

$= 6$

Learning Activity 12.1.4.1

5 minutes

Evaluate the following determinants:

1. $\begin{vmatrix} 4 & -3 \\ 3 & -5 \end{vmatrix}$

Answer:__________

2. $\begin{vmatrix} 4 & -14 \\ 3 & -5 \end{vmatrix}$

Answer:__________

3. $\begin{vmatrix} 4 & 3 \\ 2 & 7 \end{vmatrix}$

Answer:__________

4. $\begin{vmatrix} -14 & -3 \\ -5 & -5 \end{vmatrix}$

Answer:__________

5. $\begin{vmatrix} 4 & 6 \\ 8 & -3 \end{vmatrix}$

Answer:__________

6. $\begin{vmatrix} 2 & -4 \\ 5 & -3 \end{vmatrix}$

Answer:__________

12.1.4.2 Determinant Method of Solving Second Order Linear System

The general solution of a second-order linear system

Consider the system of equations:

$$a_1x + b_1y = c_1 \longrightarrow \textbf{Equation 1}$$
$$a_2x + b_2y = c_2 \longrightarrow \textbf{Equation 2}$$

We may solve for x and y using the addition–subtraction method. By multiplying the first equation by b_2 and the second equation by b_1 the equations become

$$\begin{array}{ll} a_1b_2x + b_1b_2y = b_2c_1 & \text{Eq.1} \\ \underline{a_2b_1x + b_1b_2y = b_1c_2} & \text{Eq.2} \\ (a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2 & \end{array} \quad \boxed{\text{Subtract Eq.1 from Eq. 2}}$$

Factoring out the x: $x(a_1b_2 - a_2b_1) = b_2c_1 - b_1c_2$

Solving for x: $x = \dfrac{b_2c_1 - b_1c_2}{a_1c_2 - a_2c_1}$

Similarly, we find $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$ when $a_1b_2 - a_2b_1 \neq 0$.

The expression $a_1b_2 - a_2b_1$ is often written in the form $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ called **determinant of the second order.**

The numbers a_1, a_2, b_1, b_2 are the elements of the determinant while the elements a_1 and b_2 form the principal diagonal. Therefore the expressions for x and y can be written as

$$x = \frac{b_2c_1 - b_1c_2}{a_1c_2 - a_2c_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

where: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Note:

The determinants in the denominators are the same and they are formed from the coefficients of x and y in the original equations.

The first column in the denominator consists of the respective coefficients of x, and the second column consists of the respective coefficient of y in the given simultaneous equations.

The numerator of the value of x which is

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

is the same as the denominator, only the first column is replaced by the constant terms in the original equations. Similarly the second column in the numerator of y is replaced by the constant terms.

Names associated with a linear system of equation

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

Determinant of coefficients

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{matrix} c_1 \\ c_2 \end{matrix}$$

Column of constants

Column of y - coefficients

Column of x- coefficients

Now let us look at some examples.

Example 1

Solve the following system of equation using determinants.

$$x + 4y = 7$$

$$3x + 5y = 0$$

Solution:

Compared with the equations in 1 and 2 in page 67, we find that:

$a_1 = 1; b_1 = 4; c_1 = 7; a_2 = 3; b_2 = 5$ and $c_2 = 0$

Hence:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} 7 & 4 \\ 0 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix}} = \frac{(7)(5)-(0)(4)}{(1)(5)-(3)(4)} = \frac{35}{-7} = \mathbf{-5}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 7 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix}} = \frac{(1)(0)-(3)(7)}{(1)(5)-(3)(4)} = \frac{-21}{-7} = \mathbf{3}$$

Therefore, the solution is x = -5 and y = 3, written (-5, 3).

This is a consistent system made up of independent equations.

To check if the answer is correct, substitute x = -5 and y = 3 in any of the two equations.

Hence,

$$\begin{aligned} x + 4y &= 7 \\ -5 + 4(3) &= 7 \\ -5 + 12 &= 7 \\ 7 &= 7 \end{aligned} \qquad\qquad \begin{aligned} 3x + 5y &= 0 \\ 3(-5) + 5(3) &= 0 \\ -15 + 15 &= 0 \\ 0 &= 0 \end{aligned}$$

The values obtained for x and for y satisfy each of the original equations in the system; therefore, the solution is correct.

Example 2

Solve by determinants:

$$x + 2y = 8$$
$$3x + y = 9$$

Solution: $a_1 = 1$; $b_1 = 2$; $c_1 = 8$,

$a_2 = 3$; $b_2 = 1$; $c_2 = 9$

Hence,

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} 8 & 2 \\ 9 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \frac{(8)(1)-(9)(2)}{(1)(1)-(3)(2)} = \frac{-10}{-5} = 2$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 8 \\ 3 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \frac{(1)(9)-(3)(8)}{(1)(1)-(3)(2)} = \frac{-15}{-5} = \mathbf{3}$$

Therefore, the solution is x = 2 and y = 3, written (2, 3).

Check:

$x + 2y = 8$	$3x + y = 9$
$2 + 2(3) = 8$	$3(2) + 3 = 9$
$2 + 6 = 8$	$6 + 3 = 9$
$8 = 8$	$9 = 9$

The values obtained for x and for y satisfy each of the original equations in the system, therefore the solution is correct. The system is consistent.

Example 3

Solve by determinants: $4x + 3y = 2$

$3x + 5y = -4$

Solution: $a_1 = 4$; $b_1 = 3$; $c_1 = 2$

$a_2 = 3$; $b_2 = 5$; $c_2 = -4$

Hence,

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} 2 & 3 \\ -4 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix}} = \frac{(2)(5)-(-4)(3)}{(4)(5)-(3)(3)} = \frac{22}{11} = 2$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} 4 & 2 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix}} = \frac{(4)(-4)-(3)(2)}{(4)(5)-(3(3))} = \frac{-22}{11} = -2$$

Therefore, the solution is x = 2 and y = -2, written (2, -2).

Check:

$4(2) + 3(-2) = 2$	$3(2) + 5(-2) = -4$
$8 - 6 = 2$	$6 - 10 = -4$
$2 = 2$	$-4 = -4$

The values obtained for x and for y satisfy each of the original equations in the system, therefore the solution is correct. The system is consistent.

Learning Activity 12.1.4.2

5 minutes

Solve the following systems of equations by determinants and check.

1. $2x + y = 7$
   $x + 2y = 8$

ANSWER:______________

2. $x - 2y = 3$
   $3x + 7y = -4$

ANSWER:______________

3. $x + 3y = 1$
   $2x + 6y = 3$

ANSWER:______________

4. $x + 2y = -1$
   $2x - 3y = 12$

ANSWER:______________

5. $3x + 2y = 7$
   $-x + 4y = 0$

ANSWER:______________

12.1.4.3 The Cramer's Rule

The Cramer's Rule is another method that can solve systems of linear equations using determinants.

The linear system $a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

has the solution

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

provided $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

According to the Cramer's Rule, the above pair of equation can be solved by simply calculating three determinants. One is called the **denominator determinant**, labeled **D**; another is the **x numerator determinant**, labeled $\mathbf{D_x}$; and the third is the **y numerator determinant**, labeled $\mathbf{D_y}$.

Using the notation, we can formulate the rule.

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

We can write the solution of the system as

$$\boxed{X = \frac{|D_X|}{|D|} \quad \text{and} \quad y = \frac{|D_Y|}{|D|}}$$

Points to consider when looking at the formula

1) The coefficients of variables **x** and **y** use subscripted **a** and **b**, respectively. While the constant terms use subscripted **c**.
2) Both denominators in solving **x** and **y** are the same. They come from the columns of **x** and **y**.
3) Looking at the numerator in solving for **x**, the coefficients of x-column are replaced by the constant column.

4) In the same manner, to solve for **y**, the coefficients of y-column are replaced by the constant column.

Now look at the examples of using the Cramer's Rule to solve system of equations with two variables.

Example 1

Use the Cramer's rule to solve the system: $x + 4y = 7$

$$3x + 5y = 0$$

Solution: $a_1 = 1$; $b_1 = 4$; $c_1 = 7$
$a_2 = 3$; $b_2 = 5$; $c_2 = 0$

To solve the system, find the three determinants that are created. The first is called the **denominator determinant**, labeled **D**; the second is the **x numerator determinant**, labeled **D_x**; and the third is the **y numerator determinant**, labeled **D_y**.

(a) The denominator determinant, **D**, is formed by taking the coefficients of x and y from the equation written in standard form.

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix} = (1)(5) - (3)(4)$$

$$= 5 - 12$$

$$\mathbf{= -7}$$

(b) The x-numerator determinant, **D_x**, is formed by taking the constant terms from the system and placing them in the y-coefficient positions and retaining the x-coefficients.

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \begin{vmatrix} 7 & 4 \\ 0 & 5 \end{vmatrix} = (7)(5) - (0)(4)$$

$$= 35 - 0$$

$$\mathbf{= 35}$$

(c) The y-numerator determinant, **D_y**, is formed by taking the constant terms from the system and placing them in the x-coefficient positions and retaining the y-coefficients.

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} 1 & 7 \\ 3 & 0 \end{vmatrix} = (1)(0) - (3)(7)$$

$$= 0 - 21$$

$$\mathbf{= -21}$$

(d) To find for x, we have:

$$x = \frac{D_x}{D} = \frac{35}{-7} = -5$$

(e) To find for y, we have:

$$y = \frac{D_y}{D} = \frac{-21}{-7} = 3$$

Therefore, the solution of the system is (-5, 3).

To check if the answer is correct, substitute x = -5 and y = 3 in any of the two equations.

Hence,

$x + 4y = 7$	$3x + 5y = 0$
$-5 + 4(3) = 7$	$3(-5) + 5(3) = 0$
$-5 + 12 = 7$	$-15 + 15 = 0$
$7 = 7$	$0 = 0$

The values for x and y satisfy the two equations, so the solution is correct and the system is consistent.

Example 2

Solve the system of equation below using the Cramer's Rule.

$$4x - 3y = -14$$
$$3x - 5y = -5$$

Solution: $a_1 = 4$; $b_1 = -3$; $c_1 = -14$
$a_2 = 3$; $b_2 = -5$; $c_2 = -5$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ 3 & -5 \end{vmatrix} = (4)(-5) - (3)(-3)$$
$$= -20 + 9$$
$$= \mathbf{-11}$$

$$D_X = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \begin{vmatrix} -14 & -3 \\ -5 & -5 \end{vmatrix} = (-14)(-5) - (-3)(-5)$$
$$= 70 - 15$$
$$= \mathbf{55}$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & -14 \\ 3 & -5 \end{vmatrix} = (4)(-5) - (3)(-14)$$

$$= -20 + 42$$

$$= \mathbf{22}$$

To find for x, we have:

$$x = \frac{D_x}{D} = \frac{55}{-11} = \mathbf{-5}$$

To find for y, we have:

$$y = \frac{D_y}{D} = \frac{22}{-11} = \mathbf{-2}$$

Therefore, the solution of the system is (-5, -2)

To check if the answer is correct, substitute x = -5 and y = -2 in any of the two equation.

Hence,	$4x - 3y = -14$	$3x - 5y = -5$
	$4(-5) - 3(-2) = -14$	$3(-5) - 5(-2) = -5$
	$-20 + 6 = -14$	$-15 + 10 = -5$
	$-14 = -14$	$-5 = -5$

The values for x and y satisfy the two equations, so the solution is correct and the system is consistent.

Example 3

By using the Cramer's Rule, solve the system $3x + 5y = 15$

$6X + 10y = 30$

Solution: $a_1 = 3$; $b_1 = 5$; $c_1 = 15$
$a_2 = 6$; $b_2 = 10$; $c_2 = 30$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 6 & 10 \end{vmatrix} = (3)(10) - (3)(10)$$

$$= 30 - 30$$

$$= 0$$

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \begin{vmatrix} 15 & 5 \\ 30 & 10 \end{vmatrix} = (15)(10) - (30)(5)$$

$$= 150 - 150$$

$$= 0$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} 3 & 15 \\ 6 & 30 \end{vmatrix} = (3)(30) - (6)(15)$$

$$= 90 - 90$$

$$= \mathbf{0}$$

To find for x, we have:

$$x = \frac{D_x}{D} = \frac{0}{0} \text{ cannot determined}$$

To find for y, we have:

$$y = \frac{D_y}{D} = \frac{0}{0} \text{ cannot be determined}$$

Therefore, the solution of the system cannot be found by the Cramer's Rule. This is a consistent system made up of dependent equations. There are many solutions.

The method for solving a second-order linear system by using the Cramer's rule can be summarised as follows:

To solve the linear system $\begin{Bmatrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{Bmatrix}$ by using the Cramer's Rule

The solution is $x = \dfrac{D_x}{D}$; $y = \dfrac{D_y}{D}$

Where: $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ Determinant of coefficients

$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ column of constants in place of x- coefficients

$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ Column of constants in place of y-coefficients

There are three possibilities when solving a system of linear equations by using the Cramer's Rule.

1. Only one solution (Consistent system): the determinant of the coefficients is not equal to zero.

2. No solution (Inconsistent system): the determinant of the coefficient is equal to zero, and the determinants serving as numerators for the letters are not equal to zero.

3. Many solutions (Dependent equations): the determinants of the coefficients are equal to zero, and the determinants serving as numerators for the letters are also equal to zero.

Learning Activity 12.1.4.3 5 minutes

1. Find the value of each second-order determinants.

a. $\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}$

d. $\begin{vmatrix} 5 & 0 \\ -9 & 8 \end{vmatrix}$

b. $\begin{vmatrix} 4 & 3 \\ 2 & 7 \end{vmatrix}$

e. $\begin{vmatrix} -7 & -3 \\ 5 & 8 \end{vmatrix}$

c. $\begin{vmatrix} 2 & -4 \\ 5 & -3 \end{vmatrix}$

f. $\begin{vmatrix} 2 & -4 \\ -3 & 6 \end{vmatrix}$

2. Use the Cramer's Rule to solve each system, write "inconsistent" if no solution exists. Write "dependent" if an infinite number of solution exists.

a. $2x + y = 7$
$x + 2y = 8$

b. $x + 3y = 6$
$2x + y = 7$

c. $x - 2y = 3$
$3x + 7y = -4$

d. $5x + 7y = 1$
$3x + 4y = 1$

e. $3x - 4y = 5$
$9x ++ 8y = 0$

f. $x + 3y = 1$
$2x + 6y = 3$

g. $2x - 4y = 6$
$3x - 6y = 9$

NOW DO THE SUMMATIVE TASK 4

SUMMATIVE TASK 12.1.4

30 minutes

1. Write the correct term or terms on the spaces provided.

   a. Determinant is a ___________of numbers or variables enclosed between vertical lines.

   b. The ___________ of a determinant are the numbers in its array.

   c. A ___________ of a determinant is a horizontal line of its elements.

   d. A ___________ of a determinant is a vertical line of its elements.

   e. The ___________of a determinant is the line of its elements from upper left corner to the lower right corner.

   f. The ___________ of a determinant is the line of its elements from the lower left corner to the upper right corner.

2. Find the value of each second-order determinant.

   a. $\begin{vmatrix} -5 & 3 \\ 4 & 2 \end{vmatrix}$

   e. $\begin{vmatrix} -1 & 8 \\ 5 & 0 \end{vmatrix}$

   b. $\begin{vmatrix} 0 & 6 \\ -8 & 0 \end{vmatrix}$

   f. $\begin{vmatrix} -5 & 0 \\ 2 & 10 \end{vmatrix}$

   c. $\begin{vmatrix} 2 & -2 \\ 7 & -7 \end{vmatrix}$

   g. $\begin{vmatrix} 8 & -6 \\ -10 & 9 \end{vmatrix}$

   d. $\begin{vmatrix} -9 & -9 \\ -7 & -10 \end{vmatrix}$

   h. $\begin{vmatrix} 10 & -9 \\ -7 & 3 \end{vmatrix}$

3. Evaluate:

a. $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} -5 & -6 \\ 2 & 3 \end{vmatrix}$

b. $\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 3 & 6 \end{vmatrix}$

4. Solve the system of each of the following system of equations using determinants.

a. $5x + y = -11$
$2x + 4y = -10$

b. $3x - 5y = -31$
$2x + y = 1$

c. $5x + 2y = 1$
$7x - 6y = 8$

5. Solve each of the following the system of equations using the Cramer's rule.

a. $4x + 6y = 3$
$8x - 3y = 1$

b. $3x - 2y = 2$
$6x + 4y = 44$

c. $3m + 5n = -3$
$5m - 6n = 81$

d. $4c + 3d = 13$
$5c - 4d = 55$

UNIT SUMMARY

This summary outlines the important key ideas and concepts to be remembered.

12.1.1 Sets

- A set is a collection of objects or things. The **elements** of the set are the objects that make up the set.
- The Roster method of representing a set is a lists of its members enclosed in braces.
  { 5, 9, 7, 13}
- The modified method of representing a set is used when the number of elements in the set is so large that it is not convenient or even possible to list all it members.
  W = {0, 1, 2, 3, ...}
- The rule (or set-builder) method can also be used to represent a set.
  {x|x is an even integer}
- Equal sets are sets having exactly the same members.
  {1, 7, 1, 3} = {1, 7, 3}
- An empty set is a set having no elements. Ø and { } both read "the empty set".
- A universal set is a set consisting of all the elements being considered in a particular problem. **U** is read "the universal set".
- A finite set is a set in which the counting of all its elements comes to an end.
  {5, 4, 6, 2} has four elements
- An infinite set is a set in which the counting of all its elements never comes to an end.
  {1, 2, 3, ...} has an infinite number of elements
- A Venn diagram is a useful tool for understanding set concepts.

12.1.2 Sequences and Series

- A sequence of numbers is a set of numbers arranged in definite order.
- The numbers in a sequence are called **terms.**
- Sequences having last term are called **finite**, and sequences which are not finite are called **infinite**.
- The n^{th} term of a sequence is called the **general term** of a sequence.
- The indicated sum of the terms of a sequence is called a **series**.
- An **arithmetic progression** is a sequence wherein each term after the first is obtained by adding a fixed number called the **common difference** to the preceding term.
- The sum of the terms in an arithmetic progression is called an **arithmetic series**.
- A **geometric progression** is a sequence for which every term after the first is obtained by multiplying a fixed number called the **common ratio** to the preceding term.
- The Summation or Sigma Notation is a concise way to represent long sums using the Greek letter Sigma $\sum$.

12.1.3 Binomial Expansion

- The **Pascal's triangle** is a triangular array of numbers which is used to find the coefficients in a binomial expansion. The numbers in the triangle reveals that:

- The row numbers in the Pascal's triangle corresponds to the exponent of **n** in the expansion of $(a + b)^n$.
- The first and the last numbers in any row are 1. A number in the triangle is the sum of the two closest numbers in the row above it.
- Any number in the triangle s the sum of the two closest numbers in the row above it.

- The **Binomial theorem** is the formula used to calculate directly any coefficient in the binomial expansion. Below is the theorem.

$$(a + b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-1}b^2 + \ldots + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^2$$

12.1.4 Determinants

- A **determinant** is a square array of numbers or variables enclosed between vertical lines. The numbers in its array are the **elements** of a determinant. A **row** of a determinant is a horizontal line of its elements. A **column** of a determinant is a vertical line of its elements.
- The **principal diagonal** of a determinant is the line of its elements from the upper left corner to the lower right corner.
- The **secondary diagonal** of a determinant is the line of elements from the lower left corner to the upper right corner.
- The value of a second order determinant is found by finding the difference between the product of the numbers in the principal diagonal and the product of the numbers in the secondary diagonals.
- The Cramer's Rule is another method of solving system of equations using determinants.
- To solve system of equations using the Cramer's rule, calculate the three determinants: the first is called the denominator determinant (D); the second is the x numerator determinant (D_x); and the third is the y numerator determinant (D_y).

The solution of the linear system $\begin{Bmatrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{Bmatrix}$ using the Cramer's Rule is

$$x = \frac{D_x}{D}; \qquad y = \frac{D_y}{D}$$

REVISE SECTIONS 12.1.1 TO 12.1.4 THEN DO MODULE ASSESSMENT 1

ANSWERS TO LEARNING ACTIVITIES 12.1

Learning Activity 12.1.1.1

1. a. Roster method = {5 6, 7, 8, 9, 10}
   Rule method = $\{x | 4 < x \leq 10\}$

   b. Roster method = {m, a, t, h, e, i, c, s}
   Rule method = {x|x is a distinct letter of the word "mathematics"}

   c. Roster method = {10, 20, 30, 40, 50, 60, 70, 80, 90}
   Rule method = {x|x is a counting numbers < 100 and ends with zero}

2. a. finite
   b. finite
   c. infinite
   d. finite
   e. infinite

Learning Activity 12.1.1.2

1. Yes
2. All. (A = B = C = D)
3. a. Yes; $E \subset R$ b. No; $P \not\subset S$ c. No; $C \not\subset P$
4. b, c, and d
5. c
6. c
7. a. A = {1, 3, 5, 7, 9}
   b. B = {3}
8. a. A = {x|x is an even number}
   b. B = {x|x is a multiple of 5)

Learning Activity 12.1.1.3

1. a. {2, 7, 8, 9, 10}
   b. {1, 2, 3, 6, 7, 8}
   c. {4, 5, 6, 9, 10}
   d. {1, 3, 4, 5, 6, 9, 10}
   e. {1, 2, 3, 4, 5, 6, 7, 8}
   f. {4, 5}
   g. {1, 3, 6 }

h. {(4)(1), (4)(2), (4)(3), (4)(7), (4)(8), (5)(1), (4)(2), (3)(3), (5)(7), (5)(8), (9)(1), (9)(2), (9)(3), (9)(7), (9)(8), (10)(1), (10)(2), (10)(3), (10)(7), (10)(8)}

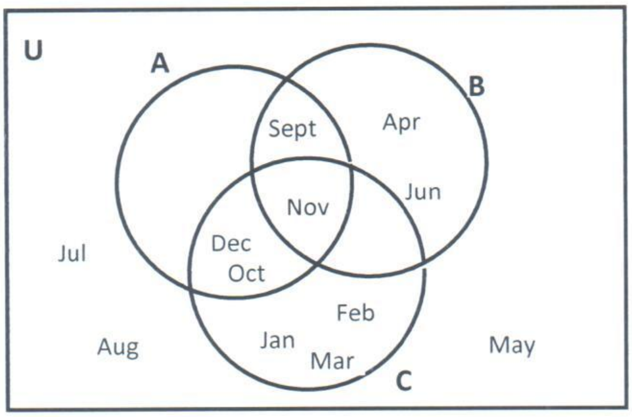
i. {(2)(1), (2)(2), (2)(3),(2)(6), (2)(7), (2)(8), (7)(1), (7)(2), (7)(3), (7)(6), (7)(7), (7)(8), (8)(1), (8)(2), (8)(3), (8)(6), (8)(7), (8)(8), (9)(1), (9)(2), (9)(3), (9)(6), (9)(7), (9)(8), (10)(1), (10)(2), (10)(3), (10)(6), (10)(7), (10)(8)}

j. {(4)(4), (4)(5), (5)(4), (5)(5), (6)(4), (6)(5)}

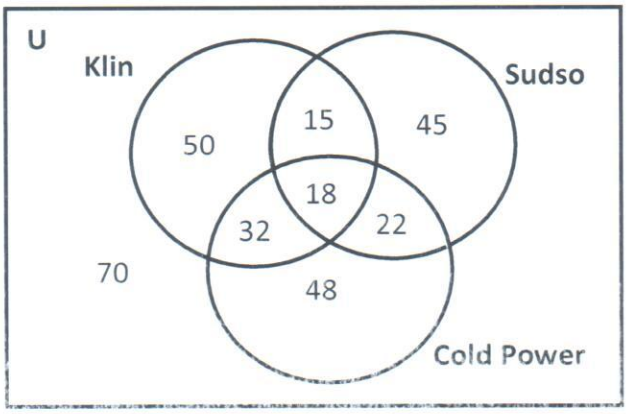
2. a. {0, 1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18}
   b. {5, 7, 11, 13, 17, 19}
   c. {1, 6, 9, 18}
   d. {2, 3} x {1, 2, 3, 4} = {(2)(1), (2)(2), (2)(3), (2)(4), (3)(1), (3)(2), (3)(3), (3)(4)}

Learning Activity 12.1.1.4

1.



a. $B \cup C$ = {Jan, Feb, Mar, Apr, Jun, Sept, Oct, Nov, Dec}
b. $B \cap C$ = {Nov}
c. $A - C$ = {Sept
2. a.

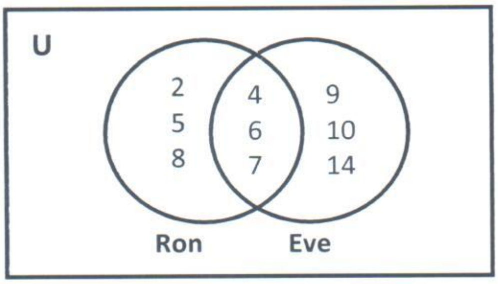


b. 18
c. 70
d. 143
e. 69

Summative Task 12.1.1

1. Yes
2. Yes
3. {o, 1, 2, 3, 4, 5, 6, 7, 8, 9}
4. a. infinite b. finite c. finite d. finite
5. a. true b. true c. false d. true
6. {2, 4, 6, 8}
7. {x|x is a number divisible by 5}
8. a. {1, 2, 3, 4, 5}

   b. {2, 4}
9. i. C ii. A iii. A iv. B v. A
10.



11. a. $A \cap B = \{3, 6\}$

    b. $A \cap C = C$

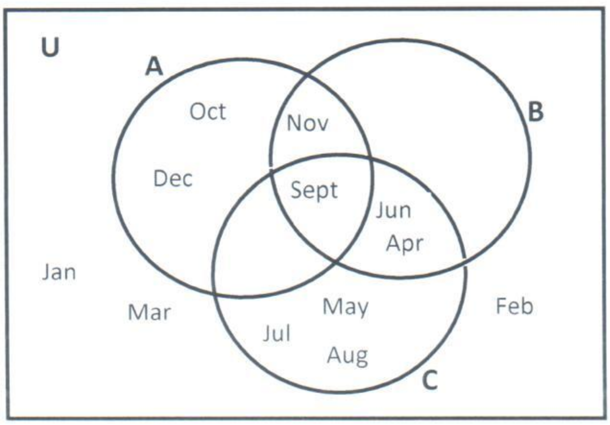
    c. $B \cap C = \emptyset$

    d. $A \cup B = \{2,3, 4, 5, 6, 8, 9, 10, 12, 15\}$

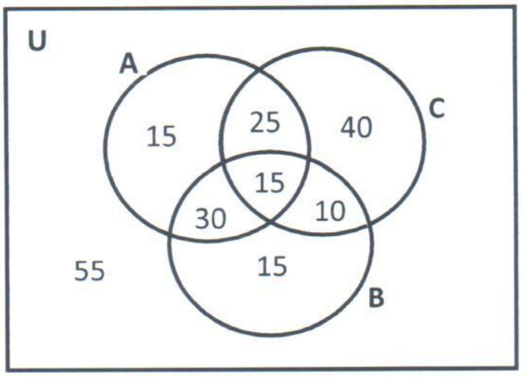
    e. $B \cup C = \{2, 3, 4, 6, 9, 12, 15$

    f. Subset of A = $C \subset A$

12. i.



ii.



Learning activity 12.1.2.1

1. a. sequence
   b. series
   c. finite sequence
   d. general term
   e. terms

2. a. finite
   b. infinite
   c. finite
   d. infinite

3. a. 3, 6, 9, 12
   b. 1, 2, 7, 14
   c. 0, 3, 8, 15
   d. 0, 2, $\frac{3}{2}$, $\frac{4}{3}$

Learning Activity 12.1.2.2

1. i. (a) adding (b) common difference
   ii. $a_n = a_1 + (n-1)d$

2. i. Yes; common difference = 3
   ii. Yes; common difference = 7
   iii. Yes; common difference = 2
   iv. Yes; common difference = 3
   v. Yes; common difference = 5

3. i. $a_5 = 18$; $S_5 = 60$

   ii. $d = 2$; $a_1 = 1$

   iii. $d = 1$; $a_1 = 2$

4. 2500

5. 22

6. 300 times

7. 3540 seats

Learning Activity 12.1.2.3

1. i. (a) multiplying (b) common ratio

   ii. $a_n = ar^{(n-1)}$

   iii. $a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^n + \ldots$

2. i. yes; $r = \frac{1}{2}$

   ii. yes; $r = 3$

3. 3, 6, 12, 24, 48

4. $r = 2$

5. K1404.93

Learning Activity 12.1.2.4

1. a. 12

   b. $\frac{77}{60}$

   c. 40

   d. 63

   e. 12

2. a. $\sum_{K=1}^{9} K^2$

   b. $\sum_{I=1}^{7} 3$

3. a. $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

   b. $2(1 + 2 + 3 + 4 + 5 + 6) = 2(21) = 42$

   c. $x + 4x^2 + + 9x^3 + 16x^4 + 25x^5 + 36x^6$

Summative task 12.1.2

A. 1. C
   2. D
   3. D
   4. A
   5. A

B. 1. -243, 729, -2187
   2. 144, 196, 256
   3. $\frac{3}{32}, \frac{7}{128}, \frac{1}{32}$
   4. 35, 48, 63

C. 1. 28, 38, 48, 58, 68
   2. 35, 39, 43, 47, 51
   3. -34, -44, -54, -64, -74

D. 1. Yes, d= -3
   2. Yes, d= -20
   3. Yes, d= -30
   4. Yes, d= 5
   5. Yes, d= -2

E. 1. Yes, r = 5

2. Yes, r = 2

3. Not Geometric

4. Yes, r = -5

5. Yes, r = -6

F. a. 9

b. 59

c. 81

G. a. K1520.875

b. 18 meters

c. K12 960

Learning Activity 12.1.3.1

1.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

2. a. $x^5 + 15x^4 + 90x^3 + 270x^2 + 810x + 243$

b. $x^5 - 15x^4 + 90x^3 - 270x^2 + 810x - 243$

c. $x^{12} + 6x^{10}y + 15x^8y^2 + 20x^6y^3 + 15x^4y^4 + 6x^2y^5 + y^6$

d. $\frac{1}{x^5} + \frac{5}{x^3} + \frac{10}{x} + 10x + 5x^3 + x^5$

e. $x^8 - 12x^6y + 54x^3y^2 - 108x^2y^3 + 81y^4$

f. $1^7 = 1$

Learning Activity 12.1.3.2

1. a. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$

   b. $x^3 + 6x^2 + 12x + 8$

2. a. 0

   b. 2

   3. $4060x^{27}b^3$

Summative Task 12.1.3

1. a. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

   b. $16x^4 + 8x^3 + 12x^2 + 8x + 16$

   c. $x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

   d. $8m^3 + 12m^2 + 18m + 27$

   e. $1 - 10x + 20x^2 - 80x^3 + 90x^4 - 32x^5$

2. a. $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

   b. $1 + 10y + 40y^2 + 80y^3 + 80y^4 + 32y^5$

   c. $243x^5 - 405x^4y^2 + 270x^3y^4 - 90x^2y^6 + 15xy^8 - y^{10}$

3. $9y^2$

4. $16b^4$

5. 252

Learning Activity 12.1.4.1

1. 11
2. 22
3. 22
4. 55
5. -60
6. 14

Learning Activity 12.1.4.2

1. (2, 3)
2. (1, -1)
3. Inconsistent; no solution
4. (3, -2)
5. $(2, \frac{1}{2})$

Learning Activity 12.1.4.3

1. a. 7
   b. 22
   c. 14
   d. 40
   e. -41
   f. 0

2. a. (2, 3)
   b. (3, 1)
   c. (1, -1)
   d. (3, -2)
   e. $(\frac{2}{3}, -\frac{3}{4})$
   f. inconsistent; no solution
   g. Dependent; an infinite solution

Summative Task 12.1.4

1. a. square array
   b. elements
   c. row
   d. column
   e. principal diagonal
   f. secondary diagonal

2. a. -22
   b. 48
   c. 0
   d. 27
   e. -40
   f. -50
   g. 12
   h. -33

3. a. 4
   b. 16

4. a. (-1, -2)
   b. (-2, 5)
   c. $(\frac{1}{2}, -\frac{3}{4})$

5. a. $(\frac{1}{4}, \frac{1}{3})$
   b. (4, -5)
   c. (-9, 6)
   d. (7, -5)

END OF UNIT MODULE 12.1

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SUBJECT AND GRADE TO STUDY

Grade Levels	Subjects
Grades 7 and 8	1. English
	2. Mathematics
	3. Science
	4. Social Science
	5. Making a Living
	6. Personal Development
	7. English
Grades 9 and 10	1. English
	2. Formal Mathematics
	3. Practical Mathematics
	4. Science
	5. Social Science
	6. Commerce
	7. Design and Technology- Computing
	8. Personal Development
Grades 11 and 12	1. English
	• Applied English • Language and Literature
	2. Mathematics
	• Mathematics A • Mathematics B
	3. Science
	4. Social Science
	• History • Geography • Economics
	5. Business Studies
	6. Personal Development
	7. ICT

REMEMBER:

- For Grades 7 and 8, you are required to do all six (6) courses.
- For Grades 9 and 10, you must study English, Mathematics, Science, Personal Development, Social Science and Commerce. Design and Technology-Computing is optional.
- For Grades 11 and 12, you are required to complete seven (7) out of thirteen (13) courses to be certified.

Your Provincial Coordinator or Supervisor will give you more information regarding each subject.

Certificate in Matriculation

CORE COURSES

Basic English
English 1
English 2
Basic Maths
Maths 1
Maths 2
History of Science & Technology

OPTIONAL COURSES

Science Streams: Biology
Chemistry, Physics and Social Science Streams: Geography, Introduction to Economics and Asia and the Modern World

REMEMBER:

You must successfully complete 8 courses; 5 compulsory and 3 optional